J. R. Schure Lc.

A COURSE IN ALGEBRA

BEING COURSE ONE IN MATHEMATICS IN THE UNIVERSITY OF WISCONSIN.

BY

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PREFACE.

The present volume originated in a desire on the part of the authors to furnish a text of Course I. as was previously mapped out by the department of mathematics at the University of Wisconsin.

The orginal intent was to produce a syllabus for the use of students in this institution, but it was subsequently thought that a work which would be useful here might also be found useful elsewhere, and hence it was decided to give the work more the character of a treatise than a syllabus. To insure the best results it was thought desirable to print the present preliminary edition and put it to the test of class room work, and at the same time to invite criticism and suggestions from teachers and others interested in mathematics, and then from the results of the authors' tests, and from the experience of others, to rewrite the work, changing it freely. For these reasons the treatment of many subjects in the following pages should be understood as merely tentative. The final form will depend entirely upon the results of experience.

An examination of the text will reveal many deviations from the beaten path, but the idea was not to deviate simply for the sake of being different from others; on the contrary the authors have freely drawn from other works. The sources from which material has been most largely drawn are the following: For problems, Christie's Test Questions and Wolstenholm's Collection; for various matters in the text, Kempt's Lehrbuch in die Moderne Algebra, and the algebras of Chrystal; Aldis; Hall and Knight; Oliver, Wait and Jones; and Todhunter; for historical notes, Marie's Histoire des Sciences Mathematiques et Physiques, and Matthiesen's Grundzuge der Antiken und Modernen Algebra.

Part II. of the present work, containing chapters on Imaginaries, the Rational Integral Function of x, Solution of Numerical Equations of Higher Degree, Graphic Representation of Equations, and Determinants, has already appeared and for this part, as well as for the present volume, suggestions are invited.

Several modifications have already suggested themselves to the authors, but it is hoped that any into whose hands either volume may fall will communicate with the authors with reference to any changes before the work is put in permanent form.

UNIVERSITY OF WISCONSIN,
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ALGEBRA.

CHAPTER I.

INTRODUCTION.

1. DEFINITIONS. When we wish to use a general term which shall inclnde in its meaning any intelligible combination of algebraic symbols and quantities, the word *Expression* will be adopted. Thus

$$(x^2-d)\ (ax^2+bx+c);\ \frac{c^2+d^2+abcd}{a^2+b^2-ac-bd};\ \sqrt{by}+\sqrt{ay}$$

may be called *expressions*. It includes the words *polynomial*, *fraction*, and *radical* and more besides.

When we wish to call attention to the fact that certain specified quantities appear in an expression it may be called a *Function* of those quantities. Thus if we desire to point out that x appears in the first expression above, it would be called a *function of* x. If we wish to state that a, b, c, and d occur in the second expression, we would call it a *function of* a, b, c, *and* d. If we wish to say that y occurs in the last expression, it may be called a *function of* y, or if we wish to say that a, b, and y occur in it, we would speak of it as a *function of* a, b, *and* y. A formal definition of the word function would be:

A Function of a quantity is a name applied to any mathematical expression in which the quantity appears.

2. DEFINITION. An expression is Integral with respect to any quantity or quantities, that is, is an integral function of those quantities, when the quantities named do not appear in any manner as divisors. Thus $5x^2+\frac{2}{3}x-\sqrt{2}$ is integral with respect to x; that is, is an integral function of x.

$$\frac{a-b+a^2b}{x^2+xy}+\frac{ab}{x}$$

is integral with respect to a and b, but fractional with respect to

x and y; that is, is an integral function of a and b, but a *fractional* function of x and y, the word fractional meaning just the opposite to integral.

3. Definition. An Expression is Rational with respect to any quantity or quantities, or is a rational function of those quantities, when the quantities referred to are not involved in any manner by the extraction of a root. Thus

$$(c+d)x^2-2\sqrt{c+d}\;x+\frac{3\sqrt[3]{c+d}}{x}$$

is rational with respect to x, but *irrational* with respect to c and d; that is, it is a rational function of x, but an irrational function of c and d, the term irrational being used in just the opposite sense from rational.

4. An expression may be both rational and integral with respect to certain quantities, in which case it may be spoken of as a Rational Integral Expression with respect to those quantities, or as a rational integral function of the quantities. In the same way we may speak of an expression which is rational and fractional with respect to certain quantities as a Rational Fractional Expression with reference to those quantities, or as a rational fractional function of the quantities. In like manner we may use the terms Irrational Integral Expression and Irrational Fractional Expression, or Irrational Integral Function or Irrational Fractional Function.

In the following examples the student is expected to answer the question, What kind of an expression? with refereuce to the quantities specified opposite each.

1. $ax^3+a^2x^2+a^3x.$ With respect to x? to a? to x and a?

2. $\dfrac{c}{2a}\left(1+\dfrac{1}{a^3}\sqrt{1-c}\right)$ With respect to a? to c?

3. $bx^2-\dfrac{a}{y^2}+xy.$ With respect to x? to y? to x and y?

4. $\dfrac{\sqrt{a}\,x^2+\sqrt[3]{b}+\sqrt[4]{c}}{ay^2+by+c}.$ With respect to x? to y? to x and y? to a, b, and c?

5. Definition. If by any operation we render an expression integral with reference to certain quantities, in respect to

which it was previously fractional, we are said to Integralize the expression with respect to those quantities. Thus the expression

$$\frac{a^2x^2}{b}+\frac{2ab}{x^2}+\frac{b^2x^2}{a}$$

is integralized with respect to a and b if it is multiplied by ab.

Similarly, if, by any operation, we render an expression rational with reference to certain quantities, in respect to which it was previously irrational, we are said to Rationalize the expression with respect to the quantities named. Thus if we square the irrational expression

$$\sqrt{x^2+\sqrt{ab}\;xy+y^2}$$

it is rationalized with respect to x and y.

6. DEFINITIONS. The Degree of a term with respect to any quantity or quantities is the sum of the exponents of the quantities named. Thus ab^2x^3y is of the third degree with reference to x, of the first degree with reference to y, of the fourth degree with reference to x and y, of the third degree with reference to a and b, etc. But the degree with reference to any quantities is not spoken of unless the term is rational and integral with respect to those quantities. Thus we do not speak of the degree of such a term as $\sqrt{\frac{a}{x^2}}$, with respect to either a or x.

The Degree of a polynomial with respect to any specified quantities is the degree of that one of its terms whose degree (with respect to the same quantities) is highest. Thus, $x^3-abx^2y^2+cxy$ is of the third degree with respect to x, of the second degree with respect to y, and of the fourth degree with respect to x and y. But the degree of a polynomial is not spoken of unless the polynomial is rational and integral with respect to the quantities specified.

It can easily be seen that the degree of the product of several polynomials is the sum of their separate degrees. Thus

$$(x^2+xy+y^2)\,(xy+bx^2y)$$

is of the fifth degree with respect to x and y; of what degree is it with respect to x? with respect to y?

The Degree of an *Equation* is the degree of the term of highest degree with respect to the *unknown* quantities. But both mem-

bers of the equation must be rational and integral with respect to the unknown quantities and the indicated operations must be performed; otherwise the degree is not spoken of.

What is the degree of

$$(x^2+y^2)(x^2y^2+1)=208x^2y^2?$$

Instead of speaking of expressions as being of the first or of the second or of the third degree, they are commonly designated by adjectives borrowed from geometry as *linear* or *quadratic* or *cubic* expressions respectively. An expression of the fourth degree is sometimes called *bi-quadratic*, meaning twice squared.

In place of the expression, "of the second degree in respect to x," it is common to say, "of the second degree *in* x."

7. DEFINITIONS. A polynomial is *Homogeneous* with respect to certain quantities when all its terms are of the same degree with respect to those quantities. Thus $a^3+a^2b+ab^2+b^3$ is homogeneous with respect to a and b.

An *equation* is Homogeneous when all the terms are of the same degree with reference to the *unknown* quantities. Thus the equation $x^2y+y^3+x^3=0$ is homogeneous, but $x^2y+y^3+x^3=20$ is not homogeneous.

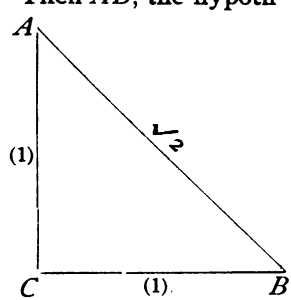
It is to be noted here that we use the term homogeneous equation in the strict sense, following the established use of the term. But in some American text books homogeneous equation includes equations like $x^2y+y^3+x^3=20$, that is, no account is taken of terms involving nothing but known quantities.

8. DEFINITION. An expression is Symmetrical with respect to two quantities if the expression is unaltered when the two quantities are interchanged. Thus x^3+y^3 is symmetrical with respect to x and y; for putting y for x and x for y we obtain y^3+x^3, which is the same as the original. Also x^2+ax+a^2 is symmetrical with respect to a and x. Is $x^2+2xy-y^2$ symmetrical with reference to x and y?

An equation of two unknown quantities is symmetrical when the interchange of the unknown quantities throughout does not modify the equation. Such is

$$x+x^2y+xy^2+y=1024.$$

9. INCOMMENSURABLE NUMBERS. Algebraic numbers* may be divided into two kinds, depending upon the relation which they bear to the unit or unity. If a number has a common measure with unity, it is called a commensurable number. Thus 7 is a commensurable number; also $\frac{3}{4}$ is a commensurable number, since one quarter of the unit is a common measure between $\frac{3}{4}$ and unity. Commensurable numbers thus include both integers and fractions. If a number has no common measure with unity, it is called an incommensurable number. Thus $\sqrt{2}$ is incommensurable. A little consideration will show that $\sqrt{2}$ cannot be an integer nor a fraction. It is not an integer because $(0)^2=0$, $(1)^2=1$, and $(2)^2=4$, and there are no integers intermediate between these. It cannot be a fraction, for if possible suppose that some irreducible fraction, represented by $\frac{a}{b}$, equals $\sqrt{2}$. Then $\sqrt{2}=\frac{a}{b}$, or squaring, $2=\frac{a^2}{b^2}$, which is absurd, for an integer cannot equal an irreducible fraction. Therefore $\sqrt{2}$ is not a fraction. But it is an *exact quantity*, for we can draw a geometrical representation of it. Take each of the two sides, CA and CB, of a right angled triangle equal to 1. Then AB, the hypothenuse, will equal $\sqrt{(1)^2+(1)^2}=\sqrt{2}$ Thus $\sqrt{2}$ is the *exact* distance from A to B, which is a perfectly definite quantity. Thus the idea that incommensurables are indefinite or inexact must be avoided. This notion has arisen because the *fractions* we often use in place of incommensurables, such as $1.4142+$ for $\sqrt{2}$, *are merely approximations* to the true value.



We now give a property of incommensurable numbers which will serve to make their separation from the class of commensurable numbers (integers and fractions) more apparent. It is that *an incommensurable number when expressed in the decimal scale never repeats, while a commensurable number so expressed always repeats.*

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*As here used the term Algebraic number does not include the so-called imaginaries, which, strictly speaking, are not numbers at all. Imaginaries are treated in Chapter I, Part II.

Thus, $75 = 75.0000000000+$ *repeating the* 0.
$\frac{1}{2} = .5000000000+$ *repeating the* 0.
$\frac{1}{3} = .3333333333+$ *repeating the* 3.
$\frac{31}{111} = .279279279279+$ *repeating the* 279.
$\sqrt{3} = 2.7320508+$ *never repeating.*
$\sqrt[3]{20} = 2.7144177+$ *never repeating.*
$\pi = 3.1415926+$ *never repeating.*

The student should endeavor to get a fair notion of what is meant by an incommensurable number. It is a difficult idea to grasp at once, but it is one which the student should continue to consider until the conception takes a definite and rational shape.

POSITIVE AND NEGATIVE QUANTITIES.

10. In Algebra we are often called upon to distinguish between quantities which are directly opposite each other; as, for instance, degrees *above* zero from degrees *below* zero on a thermometer scale, distance *north* of the Equator from distance *south* of the Equator, distance *east* of a given point from distance *west* of the same given point, etc.

The distinction is made by means of the signs + and −, *e. g.*, if $+10°$ means a temperature of 10° *above* zero, then $-10°$ would mean a temperature of 10° *below* zero, and if $+10$ miles means 10 miles *north* of the Equator, then -10 miles would mean 10 miles *south* of the Equator, and if $+10$ rods means 10 rods *east* of a given point, then -10 rods would mean 10 rods *west* of the same given point, and if $+10$ be ten units of *any* kind in *any* sense, then -10 would be ten units of the *same* kind in just the *opposite* sense.

These two kinds of quantities are called *positive* and *negative.*

11. The distinction between positive and negative quantities is made by means of the same signs as are used to denote the operations of addition and subtraction, and it might seem that it is unfortunate and unnatural that the same signs are used in these two ways. It may be unfortunate, but it is not unnatural, as we proceed to show.

12. Suppose that, by one transaction, a man gained $500, and by another he lost $700; then he lost all he gained and $200 more,

or his capital suffered a diminution of \$200. If his original capital was \$1,000, then the first transaction increased it to \$1,500, and the second transaction diminished it to \$800. Thus an addition of \$500 followed by a diminution of \$700 is equivalent to a single diminution of \$200, or

$$\$1,000+\$500-\$700=\$1,000-\$200.$$

Hence \$500—\$700 *when joined to* \$1,000 may be replaced by —\$200 *joined to* \$1,000.

Now, as any other original capital would have answered as well as \$1,000, we may neglect that original capital and write

$$\$500-\$700=-\$200.$$

Thus we see, by this illustration, that it is *natural* to prefix the minus sign to the \$200 to indicate a resultant *loss* of \$200.

13. We might have used an illustration involving some other kind of quantity than money, as *time*, *distance*, etc., and have obtained an equation similar to the one just written. We may then make an abstraction of the \$ sign and write simply

$$500-700=-200.$$

14. In Arithemetic we are concerned only with the quantities

$$0, 1, 2, 3, 4, \text{etc.},$$

and intermediate quantities, but in Algebra we consider *besides* these the quantities

$$0, -1, -2, -3, -4, \text{etc.},$$

and intermediate quantities.

15. We may represent these two classes of quantities on the following scale,

$$\ldots\ldots -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots\ldots$$

which extends indefinitely in both directions from zero.

The sign + perhaps ought to precede each of the quantities at the right of 0 in this scale, but when no sign is written before a quantity the + sign is always understood.

16. Quantities to the *right* of 0 in the above scale are *positive* and those to the *left* of 0 are *negative*, or we might say *Arabic numerals* preceded by a + sign or by no sign at all are *positive* quantities, and *Arabic numerals* preceded by a — sign are *negative* quantities.

17. In Algebra quantities are represented by letters, but a letter is just as apt to represent a quantity to the left of 0 in the above scale as it is to represent one to the right of 0; so that, while in the case of a *numerical* quantity, *i. e.* one represented by figures, we can tell whether the quantity represented is positive or negative by the sign preceding it, yet, it the case of a *literal* quantity, *i. e.* one represented by letters, we cannot tell by the sign before it whether the quantity represented is positive or negative.

If we speak of the quantity 5 we know that it is *positive*, but if we speak of the quantity a we do *not* know by the sign before it whether it is positive or negative.

We know that -5 is *negative*, but we do *not* know that $-a$ is negative.

A minus sign before a letter always represents a quantity of the opposite kind from that represented by the same quantity with a plus sign or no sign at all before it. Thus, if $a=3$, then $-a=-3$, and if $a=-3$, then $-a=3$.

18. Looking at the above scale it is evident that of any two *positive* quantities the one at the right is greater than the other or the one at the left is less than the other, *e. g.* $10>6$ or $6<10$.

Now it is found convenient to extend the meaning of the words "less than" and "greater than" so that this same thing shall be true throughout the *whole scale.*

Thus we would say that

$$-5<-3 \text{ and } -2<0.$$

It should be carefully noticed that this is a *technical* use of the words "greater than" and "less than" and conforms to the *popular* use of these words only when the quantities are positive.

Of course it would be wrong to say that -2 is less than 0 if we use "less than" in the popular sense, because no quantity can be less than nothing at all, in the popular sense of "less than."

In objecting to the use of the words "less than" in the popular sense, Prof. De Morgan, one of the great mathematicians of England, says: "The student should reject the definition still sometimes given of a negative quantity that it is less than nothing. It is astonishing that the human intellect should ever have tolerated such an absurdity as the idea of a quantity less than nothing;

above all, that the notion should have outlived the belief in judicial astrology, and the existence of witches, either of which is ten thousand times more possible."

This strong language is directed against the use of the words "less than" in the popular sense, but let the student keep in mind that the words are used in a *technical* sense and there will be no objection to such an inequality as $-2<0$.

Illustrations.—If we speak of temperature as indicated by a thermometer scale, then "*greater than*" means *higher* and "*less than*" means *lower*. If we speak of distance east and west and agree that distances measured east are positive, then "*greater than*" means "*east of*", and "*less than*" means "*west of*". If we agree that distances measured north are positive and those measured south are negative, then "*greater than*" means "*north of*", and "*less than*" means "*south of*", etc.

THE RULE OF SIGNS IN MULTIPLICATION AND DIVISION IN ALGEBRA.

19. If we take a and b any two *positive* quantities, it is easy to see that the notion of multiplication we get from arithmetic will enable us to deal with any case of multiplication where the *multiplier* is a *positive* quantity, for, evidently, a can be repeated b times, and so can $-a$ be repeated b times, but a *cannot* be repeated $-b$ times. *e. g.* 3, and also -3, can be repeated 5 times, but 3 *cannot* be repeated -5 times.

Thus, *when the multiplier is negative*, *multiplication has no meaning according to the arithmetical notion of multiplication*, and so we are obliged to broaden our ideas of multiplication in some way or else exclude the operation when the multiplier is negative.

20. The primary definition of multiplication is repeated addition, yet, even in arithmetic, the word outgrows its original meaning, for, by no stretch of language, can the operation of multiplying $\frac{1}{2}$ by $\frac{1}{2}$ be brought under the original definition.

According to the original definition, multiplication, in arithmetic, is intelligible so long as the multiplier is a *whole* number.

3 can be repeated 4 times, and so can $\frac{1}{2}$ be repeated 4 times but 4 *cannot* be repeated $\frac{1}{2}$ a time.

$\frac{1}{2}$ repeated 4 times is $\frac{1}{2}$ *multiplied by* 4, yet, in arithmetic, 4 multiplied by $\frac{1}{2}$ is a familiar operation.

Let us inquire how this comes to have a meaning, and how it happens that 4 multiplied by $\frac{1}{2}$ turns out to be $\frac{1}{2}$ *of* 4.

21. As long as a and b are positive *whole* numbers it is easy to see that $ab=ba$.

Suppose, to fix the ideas, that $a=3$ and $b=5$, then we may write down 5 rows of dots with three dots in each row, thus—

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}$$

and we have in all 5 times 3 dots. But we may look at vertical rows instead of horizontal ones and we see three rows with 5 dots in each row, and of course the number of dots is the same; so we may say

$$5\times 3=3\times 5.$$

Any other positive whole numbers would do as well as 3 and 5, and so if a and b are *any* positive whole numbers,

$$ab=ba,$$

i. e., in the product of two numbers, *it is indifferent which is the multiplier and which the multiplicand, so long as both numbers are integers.*

22. Now, *in arithmetic*, the operation of multiplication is so extended that *even when one of the quantities is a fraction* it shall *still* be indifferent which of the two quantities is the multiplier and which the multiplicand.

This *gives a meaning* to multiplication when the multiplier is a fraction, and thus it happens that 4 multiplied by $\frac{1}{2}$ is *taken to mean* the same as $\frac{1}{2}$ multiplied by 4.

23. In exactly the same way in algebra, the operation of multiplication is extended so that whatever numbers, positive or negative, integral or fractional, are represented by a and b we shall always have

$$ab=ba,$$

and since we know what is meant by -3 multiplied by 5, the equation $ab=ba$ *gives a meaning* to 5 multiplied by -3.

$$\therefore 5 \text{ multiplied by } -3=-15.$$

From this we are led to say that when the multiplier is negative, the product is just the opposite from what it would be if the multiplier were positive.

Therefore, if a and b are any two *positive* quantities, we may write the following four equations:

$$a.b=ab \qquad (1)$$
$$(-a).b=-ab \qquad (2)$$
$$a.(-b)=-ab \qquad (3)$$
$$(-a).(-b)=ab. \qquad (4)$$

From the 1st and 4th we conclude that *the product of two positive quantities or two negative quantities is positive*, and from the 2d and 3d, *the product of one positive and one negative quantity is negative.*

24. The four equations just written are true whether a and b are positive nor not.

Consider, for example, the second equation under the supposition that a is negative and b positive; then $(-a).b$ becomes the product of *two positive* quantities and is therefore positive, but $-ab$ is *also positive* in this case, as it should be, rendering the equation still true. And so of the other equations, whether a and b are positive or not. Therefore, directing our attention to the *signs*, we may say that the product of two quantities preceded by *like* signs is a quantity preceded by the $+$ sign, and the product of two quantities preceded by *unlike* signs is a quantity preceded by a $-$ sign.

This statement is usually shortened into the following—

In multiplication *like signs give plus and unlike signs give minus.*

This is often confused with the statement in italics in the preceding article. They are not identical, but both are true.

25. As division is the inverse of multiplication, it easily follows that the *quotient* of *two positive* or *two negative* quantities is *positive*, and that the *quotient* of a *positive* by a *negative* quantity,

or a *negative* by a *positive* quantity, is *negative.* It also follows that in division *like signs give plus* and *unlike signs give minus.*

The proof of these two statements is left as an exercise for the student.

26. THEOREM. *The difference between like powers of two quantities is exactly divisible by the difference of the quantities themselves.*

It is easily seen on trial that

$$(a^2-x^2)\div(a-x)=a+x.$$
$$(a^3-x^3)\div(a-x)=a^2+ax+x^2.$$
$$(a^4-x^4)\div(a-x)=a^3+a^2x+ax^2+x^3.$$
$$(a^5-x^5)\div(a-x)=a^4+a^3x+a^2x^2+ax^3+x^4.$$

Observing the uniform law in these results it would be at once suggested that the theorem is universally true; that is, that whatever be the value of n,

$$\frac{a^n-x^n}{a-x}=a^{n-1}+a^{n-2}x+a^{n-3}x^2+\ .\ .\ +a^2x^{n-3}+ax^{n-2}+x^{n-1}. \qquad (1)$$

This can easily be shown to be true, for multiplying the right hand side of this equation by $a-x$ it becomes a^n-x^n, as follows:

$$\begin{array}{llllllll}
 & a^{n-1}+a^{n-2}x+a^{n-3}x^2+\ .\ .\ .\ +a^2x^{n-3}+ax^{n-2}+x^{n-1} \\
 & \qquad\qquad\qquad\qquad\qquad\qquad a-x \\
\hline
a^n & +a^{n-1}x+a^{n-2}x^2+\ .\ .\ .\ +a^3x^{n-3}+a^2x^{n-2}+ax^{n-1} \\
 & -a^{n-1}x-a^{n-2}x^2-\ .\ .\ .\ -a^3x^{n-3}-a^2x^{n-2}-ax^{n-1}-x^n \\
\hline
a^n & +0 \quad +0 \quad +\ .\ .\ .\ +0 \quad +0 \quad +0 \quad -x^n
\end{array}$$

But multiplying the left hand side of equation (1) by $a-x$ we obtain a^n-x^n also. Hence equation (1) reduces to

$$a^n-x^n=a^n-x^n,$$

and hence must be correct.

27. THEOREM. *The difference of like even powers is exactly divisible by the sum of the quantities themselves.*

It will be found on actual division that

$$(a^2-x^2)\div(a+x)=a-x.$$
$$(a^4-x^4)\div(a+x)=a^3-a^2x+ax^2-x^3.$$
$$(a^6-x^6)\div(a+x)=a^5-a^4x+a^3x^2-a^2x^3+ax^4-x^5.$$

The obvious uniformity in these results forces the suggestion that the law of formation of the quotient will hold in any similar case. That is, that

$$\frac{a^n-x^n}{a+x}=a^{n-1}-a^{n-2}x+a^{n-3}x^2-\ \ldots\ -a^2x^{n-3}+ax^{n-2}-x^{n-1}, \qquad (1)$$

where we have given the $-$ sign to the odd powers of x, n being any even number. Multiplying the right hand side of the equation by $a-x$ we obtain a^n-x^n, thus:

$$\begin{array}{llllllll}
 & a^{n-1} & -a^{n-2}x & +a^{n-3}x^2- & \ldots & -a^2x^{n-3} & +ax^{n-2} & -x^{n-1} \\
 & & & & & & a+x & \\ \hline
a^n & -a^{n-1}x & +a^{n-2}x^2- & \ldots & -a^3x^{n-3} & +a^2x^{n-2} & -ax^{n-1} & \\
 & +a^{n-1}x & -a^{n-2}x^2+ & \ldots & +a^3x^{n-3} & -a^2x^{n-2} & +ax^{n-1} & -x^n \\ \hline
a^n & +0 & +0 & +\ \ldots & +0 & +0 & +0 & -x^n
\end{array}$$

But multiplying the left hand side of equation (1) by $a+x$ we obtain a^n-x^n also. Hence equation (1) must be true, since it reduces to

$$a^n-x^n=a^n-x^n.$$

28. THEOREM. *The sum of like odd powers of two quantities is exactly divisible by the sum of the quantities themselves.*

By trial we find this theorem holds in the first few cases as follows:

$$(a+x)\div(a+x)=1.$$
$$(a^3+x^3)\div(a+x)=a^2-ax+x^2.$$
$$(a^5+x^5)\div(a+x)=a^4-a^3x+a^2x^2-ax^3+x^4.$$
$$(a^7+x^7)\div(a+x)=a^6-a^5x+a^2x^4-a^3x^3+a^2x^4-ax^5+x^6.$$

The simple law in the formation of these results would naturally suggest the general truth of the theorem. That is, that

$$\frac{a^n+x^n}{a+x}=a^{n-1}-a^{n-2}x+a^{n-3}x^2-\ \ldots\ +a^2x^{n-3}-ax^{n-2}+x^{n-1}, \qquad (1)$$

where the terms containing the odd powers of x have the minus sign, n being any odd number. Multiplying this equation through by $a+x$ it becomes

$$a^n+x^n=a^n+x^n,$$

and hence must be true.

29. The last three theorems have such a variety of applications that it is important that they should be committed to memory. We suggest the following scheme for keeping them in mind:

$x-a$ divides the difference of like powers.

$x+a$ divides the $\begin{cases} \text{differenc}\mathit{e} \text{ of like } \mathit{even} \text{ powers,} \\ \text{sum of like odd powers.} \end{cases}$

The two cases which $x+a$ divides can be kept distinct from one another by noticing that the words *difference* and *even*, which go together, are the words which contain the *e*'s.

CHAPTER II.

THEORY OF INDICES.

1. By the definition of a power of a number, a^n equals the continued product of n factors each equal to a.

$$a^n = a\ a\ a\ a\ .\ .\ .\ .\ .\ \text{to } n \text{ factors},$$

n being a positive whole number.

2. To find the product of two powers of the same letter.

$$a^3 = a\ a\ a,$$
$$a^2 = a\ a,$$
$$\therefore\ a^3a^2 = a\ a\ a\ a\ a = a^5.$$

Again,

$$a^5 = a\ a\ a\ a\ a,$$
$$a^3 = a\ a\ a,$$
$$\therefore\ a^5a^3 = a\ a\ a\ a\ a\ a\ a\ a = a^8.$$

In each case it is to be noticed that the exponent of the product equals the sum of the exponents of the two factors.

In general, if n and r are *any positive whole numbers*,

$$a^n = a\ a\ a\ a\ .\ .\ .\ .\ .\ \text{to } n \text{ factors},$$
$$a^r = a\ a\ a\ .\ .\ .\ .\ .\ .\ \text{to } r \text{ factors}.$$
$$\therefore\ a^na^r = a\ a\ a\ a\ .\ .\ .\ .\ .\ \text{to } (n+r) \text{ factors} = a^{n+r}.$$

In the present chapter the formula,

$$a^na^r = a^{n+r}, \qquad (a)$$

will be referred to as formula a.

This may be expressed in words thus—

The product of two powers of a quantitiy is equal to that quantity with an exponent equal to the suu of the exponents of the two factors.

3. We may also find the product of the *same powers* of different quantities.

$$a^2b^2 = a\ a\ b\ b = (al)(ab) = (ab)^2;$$

also

$$a^3b^3 = a\ a\ a\ b\ b\ b = (ab)(ab)(ab) = (ab)^3.$$

And so in general,

$$a^nb^n = a\ a\ a\ .\ .\ .\ \text{to } n \text{ factors} \times b\ b\ b\ .\ .\ .\ \text{to } n \text{ factors},$$
$$= (ab)(ab)(ab)\ .\ .\ .\ \text{to } n \text{ factors, each of which is } ab,$$
$$= (ab)^n.$$
$$\therefore\ a^nb^n = (ab)^n.$$

4. In the case just considered we have the *same power*, but a and b may be different. In Art. 2 we had the *same letter*, but the powers may have been different. The student should not confuse these two cases.

5. The equation $a^n a^r = a^{n+r}$ may be extended. Multiplying both sides of the equation by a^p we have

$$\therefore\ a^n a^r a^p = a^{n+r} a^p.$$

By equation (a) the right hand member equals a^{n+r+p}.

$$a^n a^r a^p = a^{n+r+p},$$

and so on, evidently, for any number of factors.

Now the exponents n, r, p may all be the same.

$$\therefore\ a^n a^n = a^{2n}, \text{ or } (a^n)^2 = a^{2n},$$

$$\text{and } a^n a^n a^n = a^{3n}, \text{ or } (a^n)^3 = a^{3n},$$

and so on. Therefore, evidently,

$$(a^n)^r = a^{nr}, \qquad (b)$$

n and r being *any positive whole numbers*. This formula will be referred to as formula (b).

6. The equation $a^n b^n = (ab)^n$ may be extended. Multiplying both sides of this equation by c^n we obtain

$$a^n b^n c^n = (ab)^n c^n = (abc)^n,$$

and so on, evidently, for any number of factors. Hence, the product of the nth powers of any number of quantities is equal to the nth power of the product of those quantities.

EXAMPLES.

1. Multiply x^5 by x^4.
2. Multiply x^7 by x^3.
3. Multiply x^4 by x^5.
4. Multiply $-x^4$ by x^5.
5. Multiply $-x^4$ by $-x^5$.
6. Multiply $(-x)^4$ by $(-x)^5$.
7. Multiply x^4 by $(-x)^5$.
8. Multiply x^4 by $(-x)^4$.
9. Multiply $(\frac{1}{2})^4$ by $(\frac{1}{2})^5$.
10. Multiply $(-\frac{1}{3})^4$ by $(-\frac{1}{3})^5$.
11. Multiply $(2a)^4$ by a^4.
12. Multiply 2^4, a^4 and a^5 together.
13. Multiply $(2a)^4$ by a^5.

14. Multiply $\left(\frac{1}{x}\right)^4$ by x^4.

15. Multiply $\left(\frac{1}{a}\right)^3$, a^3 and a together.

16. Multiply $\left(\frac{1}{x}\right)^3$ by x^4.

17. Multiply x^8 by x^3.

18. Multiply x^8, 2^3 and x^3 together.

19. Multiply x^8 by $(2x)^3$.

20. Multiply 3^8, x^8, 2^3 and x^3 together.

21. Multiply $(3x)^8$ by $(2x)^3$.

22. Multiply $(3x)^3$ by $(2x)^3$.

23. Multiply $(3x)^4$ by $(2x)^3$.

24. Multiply $(3x)^5$ by $(2x)^3$.

25. Multiply $(3x)^8$ by $(2x)^3$. Compare with example *21.*

26. Multiply $(-3x)^8$ by $(2x)^3$.

27. Multiply $(x+y)^3$ by $(x+y)^4$.

28. Multiply $(x-y)^3$ by $(x-y)^4$.

29. Multiply $(x-y)^3$ by $(x+y)^3$.

30. Multiply (x^2-y^2), $(x-y)^3$ and $(x+y)^3$ together.

31. Find the value of $[a^{(n+r)}]^{n+r}$.

32. Distinguish between $a^{(n^2)}$ and $(a^n)^2$

7. To find the quotient of two powers of the same quantity.

$$a^3 \div a^2 = \frac{a^3}{a^2} = \frac{aaa}{aa} = a,$$

and
$$a^5 \div a^3 = \frac{aaaaa}{aaa} = aa = a^2.$$

In each of these two cases the quotient is seen to be a with an exponent equal to the exponent of the dividend *minus* the exponent of the divisor.

Again,
$$a^2 \div a^3 = \frac{aa}{aaa} = \frac{1}{a},$$

and
$$a^3 \div a^5 = \frac{aaa}{aaaaa} = \frac{1}{a^2}.$$

In each of these two cases the quotient is seen to be a fraction whose numerator is 1 and whose denominator is a with an exponent equal to the exponent of the divisor *minus* the exponent of the dividend.

Now, in general, if n and r are positive whole numbers and $n>r$,

$$\frac{a^n}{a^r}=\frac{aaaa\ .\ .\ .\ \text{to } n \text{ factors,}}{aaa\ .\ .\ .\ .\ \text{to } r \text{ factors,}}$$

and the r a's in the denominator will cancel r of the a's in the numerator and leave $(n-r)$ a's *in the numerator.*

$$\therefore\ a^n \div a^r = a^{n-r}. \qquad (c)$$

This formula will be referred to as formula *(c).*

Again, n and r still being positive whole numbers, if $n<r$, we have, as before,

$$\frac{a^n}{a^r}=\frac{aaaa\ .\ .\ .\ \text{to } n \text{ factors,}}{aaa\ .\ .\ .\ .\ \text{to } r \text{ factors,}}$$

but here the n a's in the numerator will cancel n of the a's in the denominator and leave $(r-n)$ *a's in the denominator.*

$$a^n \div a^r = \frac{1}{a^{r-n}},\ \text{when } n<r. \qquad (d)$$

This formula will be referred to as formula *(d).*

8. We may also find the quotient of the *same powers* of different quantities.

$$\frac{a^2}{b^2}=\frac{aa}{bb}=\frac{a}{b}\frac{a}{b}=\left[\frac{a}{b}\right]^2$$

and

$$\frac{a^3}{b^3}=\frac{aaa}{bbb}=\frac{a}{b}\frac{a}{b}\frac{a}{b}=\left[\frac{a}{b}\right]^3,$$

and so in general,

$$\frac{a^n}{b^n}=\frac{aaa\ .\ .\ .\ \text{to } n \text{ factors,}}{bbb\ .\ .\ .\ \text{to } n \text{ factors,}}$$

$$=\frac{a}{b}\frac{a}{b}\frac{a}{b}\ .\ .\ .\ \text{to } n \text{ factors each of which equals } \frac{a}{b}$$

$$=\left[\frac{a}{b}\right]^n.$$

9. In the case just considered we have the *same power*, but the quantities a and b may be different, but in Art. 7 we have the *same quantities*, but the powers may have been different. The student should not confuse these two cases.

EXAMPLES.

1. Divide x^5 by x^3.
2. Divide x^3 by x^5.
3. Divide $(-x)^3$ by $(-x)^5$.
4. Divide $-x^3$ by $-x^5$.
5. Divide $-x^3$ by $(-x)^5$.
6. Divide $-x^4$ by $(-x)^5$.
7. Divide $(-x)^4$ by $(-x)^5$.
8. Divide $\left\{\frac{1}{m}\right\}^8$ by $\left\{\frac{1}{m}\right\}^5$
9. Divide $\left\{\frac{1}{m}\right\}^8$ by $\left\{\frac{1}{n}\right\}^8$
10. Divide $(x-y)^6$ by $\left\{\frac{1}{x-y}\right\}^6$
11. Divide $(x-y)^3$ by $\left\{\frac{1}{x+y}\right\}^3$
12. Divide $(x^2-y^2)^3$ by $(x-y)^3$.
13. Multiply b^6 by b^4 and divide the product by b^2.
14. Divide b^6 by b^2 and multiply the quotient by b^4.
15. Divide b^4 by b^2 and multiply the quotient by b^6.
16. Multiply b^6 by b^2 and divide the product by b^4.
17. Divide b^5 by b^4 and multiply the quotient by b^2.
18. Divide b^2 by b^4 and multiply the quotient by b^6.
19. Multiply b^4 by b^3 and divide the product by b^8.
20. Divide b^4 by b^8 and multiply the quotient by b^3.
21. Divide b^3 by b^8 and multiply the quotient by b^4.
22. Divide c^7 by c^2, the quotient by c^2 and so on until five divisions are performed.
23. Divide c^6 by c^2, the quotient by c^2, and so on until five divisions are performed.

10. We saw that $a^n \div a^r = a^{n-r}$ if $n > r$. Now let us take $r=1$, then (n being a positive whole number) when we divide a^n by a we simply subtract one from the exponent.

Now let us take some number for n, say 5, and divide a^5 by a, the quotient by a, and so on as long as we can, each time performing the division by subtracting one from the exponent.

We obtain the following equations:

$$a^5 \div a = a^4.$$
$$a^4 \div a = a^3.$$
$$a^3 \div a = a^2.$$
$$a^2 \div a = a.$$

If we *attempt* to go one step further by the same rule, viz: subtract one from the exponent, we get

$$a \div a = a^0.$$

Now, a^0 is a symbol that has not been used before, and indeed one that has no meaning according to the definition already given of a power of a number. But we know that $a \div a = 1$, and if we agree that this new symbol a^0 shall be 1, a being any number whatever (not zero), then we may carry our process of successive division one step further than we could without this agreement.

More than this, it may easily be seen that by giving this meaning to a^0 each of our formulas *(a)*, *(b)*, *(c)*, *(d)* is slightly more general than it was before. Let us examine these formulas separately.

11. First, $$a^n a^r = a^{n+r}.$$
If we here make $n = 0$, we get

$$a^0 a^r = a^r,$$

and this is true if $a^0 = 1$.

Again, if we make $r = 0$, we get

$$a^n a^0 = a^n,$$

and this is true if $a^0 = 1$.

12. Second, $$(a^n)^r = a^{nr}. \qquad (b)$$
If we here make $n = 0$, we get

$$(a^0)^r = a^0,$$

which is true if $a^0 = 1$, since $1^r = 1$; and if we make $r = 0$, we get

$$(a^n)^0 = a^0, *$$

and this, too, is true if *any* quantity affected with a zero exponent equals one.

---

* The student may think that such an equation as

$$4^0 = 2^0,$$

involves the absurdity that $4 = 2$, but it does not. No one thinks that the equation $\frac{4}{4} = \frac{2}{2}$ involves any absurdity, and so if we look upon a quantity affected with an exponent zero as only another way of writing a quantity divided by itself, there is no confusion.

13. Third, $a^n \div a^r = a^{n-r}$. (c)

If we make $n=r$ we get

$$a^n \div a^n = a^0,$$

and this is true if $a^0 = 1$.

Again, if we make $r=0$ we get

$$a^n \div a^0 = a^n,$$

and this also is true if $a^0 = 1$.

14. Fourth, $a^n \div a^r = \frac{1}{a^{r-n}}$. (d)

If we here make $n=r$ we get

$$a^n \div a^n = \frac{1}{a^0},$$

and this is true if $a^0 = 1$.

Again, if $n=0$ we get $a^0 \div a^r = \frac{1}{a^r}$,

and this, too, is true if $a^0 = 1$.

15. Now, because the assumption $a^0 = 1$ leads to no inconsistency it is *permissible*, and because it gives greater generality to our formulas it is *advantageous*.

Therefore we adopt the equation $a^0 = 1$ as defining the meaning of a^0.

16. The question naturally arises, is there any way whereby we may give still greater generality to our formulas?

Let us look again at our process of successive division.

We have already obtained the equations,

$$a^5 \div a = a^4,$$
$$a^4 \div a = a^3,$$
$$a^3 \div a = a^2,$$
$$a^2 \div a = a,$$
$$a \div a = a^0 = 1.$$

Now, if, by the same rule, (viz: subtract one each time from the exponent,) we attempt to take another step we get

$$a^0 \div a = a^{-1}.$$

Here, again, we have a symbol a^{-1} that has not been used be-

fore, and one which has no meaning according to the definition of a power of a number. But a^0 being 1, we know that

$$a^0 \div a = \frac{1}{a},$$

and so the equation,

$$a^0 \div a = a^{-1}$$

would be true if a^{-1} were equal to $\frac{1}{a}$.

If we could take one step in this way we ought to be able to take two or three or, indeed, any number, and if we could do this we could get, in addition to the above, the following equations:

$$a^0 \div a = a^{-1},$$
$$a^{-1} \div a = a^{-2},$$
$$a^{-2} \div a = a^{-3},$$
etc.

As we have seen, the *first* of these equations would be true if $a^{-1} = \frac{1}{a}$, and from this the *second* would be true if $a^{-2} = \frac{1}{a^2}$, and *from this* the *third* would be true if $a^{-3} = \frac{1}{a^3}$, etc., and the set of equations just written might be carried just as far as we please if

$$a^{-q} = \frac{1}{a^q},$$

q being any whole number.

Let us examine the effect of this supposition on our formulas *(a)*, *(b)*, *(c)*, *(d)*.

17. If we wish the quotient $a^n \div a^r$ we are directed in Art. 7 to use formula *(c)* if $n > r$, and formula *(d)* if $n < r$.

Suppose $n < r$ and $r - n = q$, then formula *(d)* gives

$$a^n \div a^r = \frac{1}{a^q}.$$

But if we should *try* to use formula *(c)* in this case we would get

$$a^n \div a^r = a^{-q},$$

and this would be true if

$$a^{-q} = \frac{1}{a^q},$$

so that, if we could use a quantity with a negative exponent, then

formula *(c)* could be used when $n<r$ as well as when $n>r$, and, if we like, we might retain formula *(c)* and entirely dispense with formula *(d)*.

Again, it may be seen, in a similar manner, that if $a^{-q}=\frac{1}{a^q}$ formula *(d)* could be used when $n>r$ as well as when $n<r$, so that we might, if we like, retain formula *(d)* and entirely dispense with formula *(c)*.

If we find that we may use negative exponents upon the above interpretation, then we will for the most part dispense with formula *(d)*, using it only now and then, if at all, when it comes a little handier than formula *(c)*.

18. Again, by the above interpretation formula *(c)* can be used when one or both of the exponents are negative.

First, suppose r negative and equal to $-q$, then

$$a^n \div a^{-q}=a^n \div \frac{1}{a^q}=a^n a^q=a^{n+q}.$$

But substituting in *(c)*,

$$a^n \div a^{-q}=a^{n-(-q)}=a^{n+q},$$

the same result as before, so that formula *(c)* may be used when r is negative.

Second, suppose n negative and equal to $-s$, then

$$a^{-s}\div a^r=\frac{1}{a^s}\div a^r=\frac{1}{a^s}\,\frac{1}{a^r}=\frac{1}{a^s a^r}=\frac{1}{a^{s+r}}.$$

But by substituting in *(c)*,

$$a^{-s}\div a^r=a^{-s-r}=a^{-(s+r)},$$

the same result as before if our interpretation of negative exponents be correct, so that formula *(c)* may be used when n is negative.

Third, suppose both n and r negative and let $n=-s$ and $r=-q$, then

$$a^{-s}\div a^{-q}=\frac{1}{a^s}\div\frac{1}{a^q}=\frac{a^q}{a^s}=a^{q-s}.$$

But by substituting in the formula,

$$a^{-s}\div a^{-q}=a^{-s-(-q)}=a^{q-s},$$

the same result as before, hence formula *(c)* may be used when both n and r are negative.

19. Formula *(a)* may be used when either or both of the exponents are negative if the above interpretation be correct.

First, suppose $r=-q$, then

$$a^n a^{-q}=a^n\frac{1}{a^q}=a^n\div a^q=a^{n-q}.$$

But by substituting in the formula,

$$a^n a^{-q}=a^{n-q},$$

Second, suppose $n=-q$, then

$$a^{-q}a^r=\frac{1}{a^q}a^r=a^r\div a^q=a^{r-q}.$$

But by substituting in the formula,

$$a^{-q}a^r=a^{-q+r}=a^{r-q},$$

so that formula *(a)* may be used when n is negative.

Third, suppose $n=-s$ and $r=-q$, then

$$a^{-s}a^{-q}=\frac{1}{a^s}\,\frac{1}{a^q}=\frac{1}{a^s a^q}=\frac{1}{a^{s+q}}.$$

But by substituting in the formula,

$$a^{-s}a^{-q}=a^{-s-q}=a^{-(s+q)}=\frac{1}{a^{s+q}},$$

so that formula *(a)* may be used where both n and r are negative.

20. Formula *(b)* may be used when either n or r or both are negative.

First, suppose r negative and equal to $-q$, then

$$(a^n)^{-q}=\frac{1}{(a^n)^q}=\frac{1}{a^{nq}}.$$

But by substitution in the formula,

$$(a^n)^{-q}=a^{-nq}=\frac{1}{a^{nq}},$$

so that formula *(b)* may be used when r is negative.

Second, suppose n negative and equal to $-q$, then

$$(a^{-q})^r=\left[\frac{1}{a^q}\right]^r=\frac{1}{(a^q)^r}=\frac{1}{a^{qr}},$$

But by substituting in the formula,

$$(a^{-q})^r=a^{-qr}=\frac{1}{a^{qr}},$$

so that formula *(b)* may be used when n is negative.

Third, suppose both exponents are negative and let $n=-s$ and $r=-q$, then

$$(a^{-s})^{-q}=\frac{1}{(a^{-s})^q}=\frac{1}{\left[\frac{1}{a^s}\right]^q}=\frac{1}{\frac{1}{a^{sq}}}=a^{sq}.$$

But by substituting in the formula,

$$(a^{-s})^{-q}=a^{sq},$$

so that formula *(b)* may be used when both exponents are negative.

21. Thus we see that if we interpret a^{-q} as being $\frac{1}{a^q}$, a being any number whatever (not zero), and q being any whole number, the exponents in all our formulas may be any whole numbers, positive or negative, and this makes our formulas considerably more general than they were before.

Now, because the supposition $a^{-q}=\frac{1}{a^q}$ leads to no inconsistency it is *permissible*, and because it gives greater generality to our formulas it is *advantageous*.

Therefore we adopt the equation $a^{-q}=\frac{1}{a^q}$ as defining the meaning of a^{-q}.

22. Since $a^{-q}=\frac{1}{a^q}$, and therefore $\frac{1}{a^{-q}}=a^q$, it follows that in any fraction any *factor* may be transferred from the numerator to the denominator, or *vice versa*, by simply changing the sign of the exponent.

Hence, if in formula *(c)* we transfer a^r from the denominator to the numerator, $\frac{a^n}{a^r}$ becomes $a^n a^{-r}$, which by formula *(a)* equals a^{n-r}; so that formulas *(a)* and *(c)* are really identical, but, for the sake of convenience, both are retained.

EXAMPLES.

1. Write $\frac{ab^2}{c^3d^4}$ in one line.

2. Write $\frac{3x^ty^n}{z^{-p}w^q}$ in one line.

3. Write $\frac{x^2y^3(a-b)^s}{\frac{1}{z^n}w^r}$ in one line.

4. Write $\frac{x^{-4}y^2}{5z^nw^{-r}}$ all in the *lower* line.

5. Write $\frac{4x^{-4}y^{-r}z^n}{3n^{-s}w^t}$ so that all exponents are preceded by the + sign.

6. Write $\frac{4x^{-5}y^{-6}}{u^4z^{-7}}$ with all positive exponents.

23. Having now dispensed with formula *(d)* and extended formulas *(a)*, *(b)*, *(c)* so that the exponents may be *any whole numbers*, *positive or negative*, the question arises, can we give still greater generality to our formulas by using exponents which are fractions?

24. If quantities with fractional exponents have any meaning and if we can use them in our formulas, we must have by formula *(b)*

$$\left(a^{\frac{1}{n}}\right)^n = a,$$

n being here *any positive whole number*; *i. e.* $a^{\frac{1}{n}}$ is a quantity which raised to the nth power equals a.

$$\therefore\ a^{\frac{1}{n}} = \sqrt[n]{a}.$$

Raise both sides of this equation to the rth power, r being a positive whole number, and we get

$$\left(a^{\frac{1}{n}}\right)^r = (\sqrt[n]{a})^r$$

so that, if we are permitted to use fractional exponents, $a^{\frac{r}{n}}$ denotes the rth power of the nth root of a.

25. Again, by the definition of a quantity with an exponent -1,

$$\left(a^{-\frac{r}{n}}\right)^{-1} = \frac{1}{a^{-\frac{r}{n}}},$$

and by formula *(b)*

$$\left(a^{-\frac{r}{n}}\right)^{-1} a^{\frac{r}{n}},$$

$$\therefore \frac{1}{a^{-\frac{r}{n}}} = a^{\frac{r}{r}},$$

or, taking the reciprocal of both sides,

$$a^{-\frac{r}{n}} = \frac{1}{a^{\frac{r}{n}}}.$$

.26 Thus we have *suggestions* of meanings for both positive and negative fractional exponents, and if we introduce fractional exponents into our formulas with the meanings suggested, these formulas will be found to give consistent results, as we shall see.

27. Before substituting in our formulas it is necessary to stop and show that, with the meanings suggested, a quantity with a fractional exponent has the same value whether the exponent is in its lowest terms or not.

Let $$a^{\frac{1}{qn}} = x$$
then $$a = x^{qn} = (x^q)^n$$
$$a^{\frac{1}{n}} = x^q$$
$$a^{\frac{r}{n}} = \left[a^{\frac{1}{n}}\right]^r = x^{qr} = \left[a^{\frac{1}{qn}}\right]^{qr} = a^{\frac{qr}{qn}}$$

In a similar manner it may be shown that

$$a^{-\frac{r}{n}} = a^{-\frac{qr}{qn}}$$

28. Examination of formula *(a)*.

Let $\frac{r}{n}$ and $\frac{p}{q}$ be any two positive fractions and $-\frac{r}{n}$ and $-\frac{p}{q}$ two negative fractions. Then there are four cases to consider.

First, $a^{\frac{r}{n}}a^{\frac{p}{q}}=$ what?

Second, $a^{\frac{r}{n}}a^{-\frac{p}{q}}=$ what?

Third, $a^{-\frac{r}{n}}a^{\frac{p}{q}}=$ what?

Fourth, $a^{-\frac{r}{n}}a^{-\frac{p}{q}}=$ what?

First case.

$$a^{\frac{r}{n}}a^{\frac{p}{q}}=a^{\frac{rq}{nq}}a^{\frac{np}{nq}} \text{ by art. } 27.$$

$$=\left(a^{\frac{1}{nq}}\right)^{rq}\left(a^{\frac{1}{nq}}\right)^{np}=\left(a^{\frac{1}{nq}}\right)^{rq+np}$$

$$=a^{\frac{rq+np}{nq}}=a^{\frac{rq}{nq}+\frac{np}{nq}}=a^{\frac{r}{n}+\frac{p}{q}}$$

and by direct substitution in the formula we get

$$a^{\frac{r}{n}}a^{\frac{p}{q}}=a^{\frac{r}{n}+\frac{p}{q}}$$

Second case.

$$a^{\frac{r}{n}}a^{-\frac{p}{q}}=a^{\frac{rq}{nq}}a^{-\frac{np}{nq}}=\left(a^{\frac{1}{nq}}\right)^{qr}\left(a^{\frac{1}{nq}}\right)^{-np}$$

$$=\left(a^{\frac{1}{nq}}\right)^{rq-np}=a^{\frac{rq-np}{nq}}=a^{\frac{rq}{nq}}a^{-\frac{np}{nq}}=a^{\frac{r}{n}-\frac{p}{q}}$$

and by direct substitution in the formula we also get

$$a^{\frac{r}{n}}a^{-\frac{p}{q}}=a^{\frac{r}{n}-\frac{p}{q}}$$

Third case.

This is the same as the second case, only the order of the factors is changed, and therefore as in the second case the result will be the same as given by direct substitution in the formula.

Fourth case.

$$a^{-\frac{r}{n}}a^{-\frac{p}{q}}=\frac{1}{a^{\frac{r}{n}}}\,\frac{1}{a^{\frac{p}{q}}}=\frac{1}{a^{\frac{r}{n}}a^{\frac{p}{q}}}$$

$$=\frac{1}{a^{\frac{r}{n}+\frac{p}{q}}}=a^{-\left(\frac{r}{n}+\frac{p}{q}\right)}=a^{-\frac{r}{n}-\frac{p}{q}}$$

and by direct substitution in the formula we also get

$$a^{\frac{r}{n}-\frac{p}{q}}=a^{-\frac{r}{n}-\frac{p}{q}}$$

Thus we see that by using fractional exponents according to the suggestions before obtained, the result of multiplying two fractional powers of a is, in every case, in perfect accord with formula *(a)*.

29. Examination of formula *(b)*.

As before, let $\frac{r}{n}$ and $\frac{p}{q}$ be any two positive fractions, and $-\frac{r}{n}$ and $-\frac{p}{q}$ any two negative fractions. Then there are four cases to consider.

First, $\left\{a^{\frac{r}{n}}\right\}^{\frac{p}{q}}=$ what?

Second, $\left\{a^{\frac{r}{n}}\right\}^{-\frac{p}{q}}=$ what?

Third, $\left\{a^{-\frac{r}{n}}\right\}^{\frac{p}{q}}=$ what?

Fourth, $\left\{a^{-\frac{r}{n}}\right\}^{-\frac{p}{q}}=$ what?

First case.

$$\text{Let} \quad a^{\frac{1}{nq}}=x, \qquad (1)$$

$$\therefore a^{\frac{1}{n}}=x^q; \qquad (2)$$

$$\therefore a^{\frac{r}{n}}=x^{rq}=(x^r)^q; \qquad (3)$$

$$\therefore \left\{a^{\frac{r}{n}}\right\}^{\frac{1}{q}}=x^r; \qquad (4)$$

$$\therefore \left\{\left\{a^{\frac{r}{n}}\right\}^{\frac{1}{q}}\right\}^p=x^{rp}; \qquad (5)$$

$$\text{or} \quad \left\{a^{\frac{r}{n}}\right\}^{\frac{p}{q}}=x^{rp}; \qquad (6)$$

$$\text{From } (1) \; a^{\frac{rp}{nq}}=x^{rp}; \qquad (7)$$

$$\therefore \text{from } (6) \text{ and } (7) \; \left\{a^{\frac{r}{n}}\right\}^{\frac{p}{q}}=a^{\frac{rp}{nq}},$$

and by direct substitution in the formula we also get

$$\left\{a^{\frac{r}{n}}\right\}^{\frac{p}{q}}=a^{\frac{rp}{nq}}.$$

Second case.

$$\left\{a^{\frac{r}{n}}\right\}^{-\frac{p}{q}}=\frac{1}{\left\{a^{\frac{r}{n}}\right\}^{\frac{p}{q}}}=\frac{1}{a^{\frac{rp}{nq}}}=a^{-\frac{rp}{nq}},$$

and by direct substitution in the formula we also get

$$\left\{a^{\frac{r}{n}}\right\}^{-\frac{p}{q}}=a^{-\frac{rp}{nq}}.$$

Third case.

$$\left\{a^{-\frac{r}{n}}\right\}^{\frac{p}{q}}=\left\{\frac{1}{a^{\frac{r}{n}}}\right\}^{\frac{p}{q}}=\frac{1}{\left\{a^{\frac{r}{n}}\right\}^{\frac{p}{q}}}=\frac{1}{a^{\frac{rp}{nq}}}=a^{-\frac{rp}{nq}}$$

and by direct substitution in the formula we also get

$$\left(a^{-\frac{r}{n}}\right)^{\frac{p}{q}}=a^{-\frac{rp}{nq}}.$$

Fourth case.

$$\left(a^{-\frac{r}{n}}\right)^{-\frac{p}{q}}=\frac{1}{\left(a^{-\frac{r}{n}}\right)^{\frac{p}{q}}}=\frac{1}{\left(\frac{1}{a^{\frac{r}{n}}}\right)^{\frac{p}{q}}}=\left(\frac{1}{\frac{1}{a^{\frac{r}{n}}}}\right)^{\frac{p}{q}}$$

$$=\left(a^{\frac{r}{n}}\right)^{\frac{p}{q}}=a^{\frac{rp}{nq}},$$

and by direct substitution in the formula we also get

$$\left(a^{-\frac{r}{n}}\right)^{-\frac{p}{q}}=a^{\frac{rp}{nq}}.$$

Thus we see that by using fractional exponents according to the suggestion before obtained, the result of raising any fractional power of a to any other fractional power is, in every case, in perfect accord with formula *(b)*.

30. Examination of formula *(c)*.

As before, let $\frac{r}{n}$ and $\frac{p}{q}$ be any two positive fractions and $-\frac{r}{n}$ and $-\frac{p}{q}$ any two negative fractions. Then we have four cases to consider.

First, $a^{\frac{r}{n}}\div a^{\frac{p}{q}}=$ what?

Second, $a^{\frac{r}{n}}\div a^{-\frac{p}{q}}=$ what?

Third, $a^{-\frac{r}{n}}\div a^{\frac{p}{q}}=$ what?

Fourth, $a^{-\frac{r}{n}}\div a^{-\frac{p}{q}}=$ what?

First case.

$$a^{\frac{r}{n}}\div a^{\frac{p}{q}}=a^{\frac{r}{n}}a^{-\frac{p}{q}}=a^{\frac{r}{n}-\frac{p}{q}}$$

by Art. 29, second case, and by direct substitution in the formula we also get

$$a^{\frac{r}{n}}\div a^{\frac{p}{q}}=a^{\frac{r}{n}-\frac{p}{q}}.$$

Second case.

$$a^{\frac{r}{n}}\div a^{-\frac{p}{q}}=a^{\frac{r}{n}}a^{\frac{p}{q}}=a^{\frac{r}{n}+\frac{p}{q}},$$

and by direct substitution in the formula we also get

$$a^{\frac{r}{n}}\div a^{-\frac{p}{q}}=a^{\frac{r}{n}+\frac{p}{q}}.$$

Third case.

$$a^{-\frac{r}{n}} \div a^{\frac{p}{q}} = a^{-\frac{r}{n}} a^{-\frac{p}{q}} = a^{-\frac{r}{n}-\frac{p}{q}},$$

and by direct substitution in the formula we also get

$$a^{-\frac{r}{n}} \div a^{\frac{p}{q}} = a^{-\frac{r}{n}-\frac{p}{q}}.$$

Fourth case.

$$a^{-\frac{r}{n}} \div a^{-\frac{p}{q}} = a^{-\frac{r}{n}} a^{\frac{p}{q}} = a^{-\frac{r}{n}+\frac{p}{q}},$$

and by direct substitution in the formula we also get

$$a^{-\frac{r}{n}} \div a^{-\frac{p}{q}} = a^{-\frac{r}{n}+\frac{p}{q}}.$$

Thus we see that by using fractional exponents according to the suggestion before obtained, the result of dividing one fractional power of a by another fractional power of a is, in every case, in perfect accord with formula *(c)*.

31. Now, because the suppositions

$$a^{\frac{r}{n}} = (\sqrt[n]{a})^r \text{ and } a^{-\frac{r}{n}} = \frac{1}{(\sqrt[n]{a})^r}$$

lead to no inconsistency they are *admissible*, and because they give greater generality to our formulas they are *advantageous*. Therefore we adopt these equations to *define the meaning* of quantities affected with fractional exponents.

32. The formulas *(a)*, *(b)*, *(c)* are now so generalized by the above definitions that they can be used when the exponents are any positive or negative whole numbers or fractions, and it might naturally be asked, is this the greatest generality of which they are capable?

Excluding the so-called imaginaries, there is no kind of algebraic numbers not yet discussed except incommensurable numbers, and the consideration of quantities affected with incommensurable indices is reserved for chapter XI. In the meantime, however, it should be remembered that the formulas are to be used only when the indices are commensurable.

33. By means of the meanings now given to negative and fractional exponents it is easy to see that the formula

$$a^n b^n c^n \;.\;.\;.=(a\,b\,c\;.\;.\;.)^n$$

holds whether n is positive or negative, integral or fractional.

First, let $n=\frac{1}{r}$, a positive fraction, and let $a^{\frac{1}{r}}=x$ and $b^{\frac{1}{r}}=y$;

$$\therefore\; a=x^r \text{ and } b=y^r$$

$$\text{then } a^{\frac{1}{r}}b^{\frac{1}{r}}=xy,$$

$$\text{and } ab=x^r y^r=(xy)^r.$$

$$\therefore\; (ab)^{\frac{1}{r}}=xy,$$

$$\therefore\; a^{\frac{1}{r}}b^{\frac{1}{r}}=(ab)^{\frac{1}{r}}.$$

Multiply both sides by $c^{\frac{1}{r}}$ and we get

$$a^{\frac{1}{r}}b^{\frac{1}{r}}c^{\frac{1}{r}}=(a\,b)^{\frac{1}{r}}c^{\frac{1}{r}}=(abc)^{\frac{1}{r}},$$

and so on, evidently, for any number of factors.

This is quite an important formula, stated in words it is,

The product of the r th roots of several quantities equals the r th root of their product.

Second, let $n=-r$, a negative quantity, either integral or fractional, then

$$a^{-r}\,b^{-r}=\frac{1}{a^r}\;\frac{1}{b^r}=\frac{1}{(ab)^r}=(ab)^{-r}.$$

Similarly $$a^{-r}b^{-r}c^{-r}=(abc)^{-r},$$

and so on, evidently, for any number of factors.

34. The formula $\frac{a^n}{b^n}=\left\{\frac{a}{b}\right\}^n$ also holds good whether n is positive or negative, integral or fractional.

First. let $n=\frac{1}{b}$, a positive fraction, then $\frac{a^{\frac{1}{r}}}{b^{\frac{1}{r}}}=a^{\frac{1}{r}}\left\{\frac{1}{b}\right\}^{\frac{1}{r}}=\left\{\frac{a}{b}\right\}^{\frac{1}{r}}$

Stated in words, this is,

The quotient of the r th roots of two quantities equals the r th root of their quotient.

Second, let $n=-r$, a negative quantity, either integral or fractional, then

$$\frac{a^{-r}}{b^{-r}}=\frac{b^r}{a^r}=\left(\frac{b}{a}\right)^r=\left(\frac{1}{\frac{b}{a}}\right)^{-r}=\left(\frac{a}{b}\right)^{-r}$$

EXAMPLES.

1. Write the following expressions, using fractional exponents in place of the radical signs:

$$\sqrt[5]{a^4},\quad (\sqrt[5]{a})^4,\quad \sqrt[4]{b^3x^2y^2},\quad \sqrt[5]{x^2+y^3+z^4},$$

$$\frac{1}{\sqrt[3]{a^2x}},\quad \sqrt[3]{\frac{1}{b^2y}},\quad \sqrt[6]{\frac{x^4y^2}{z^8}}.$$

2. Write the following expressions, using radical signs in place of fractional exponents:

$$x^{\frac{3}{5}},\quad (x^2+2xy+y^2)^{\frac{1}{4}},\quad (x^2+y)^{\frac{1}{2}},\quad \left(x^{\frac{1}{2}}+y^{\frac{1}{3}}\right)^{\frac{2}{5}},\quad (x^py^q-z^n)^{\frac{p}{q}}.$$

3. Multiply $x^{-\frac{5}{4}}$ by $x^{\frac{5}{2}}$.

4. Multiply $x^{\frac{2}{3}}$ by $x^{-\frac{3}{2}}$.

5. Multiply $\dfrac{1}{x^{\frac{2}{3}}}$ by $\dfrac{1}{x^{-\frac{2}{3}}}$.

6. Multiply $\left\{\dfrac{1}{x}\right\}^{-\frac{2}{3}}$ by $\dfrac{1}{x^{-\frac{3}{2}}}$.

7. Divide $(x+y)^{\frac{9}{10}}$ by $(x+y)^{\frac{19}{10}}$.

8. Divide $x^{\frac{2}{3}}-xy^{\frac{1}{2}}+x^{\frac{1}{2}}y-y^{\frac{3}{2}}$ by $x^{\frac{1}{2}}-y^{\frac{1}{2}}$.

9. Divide $x^{\frac{3}{2}}+y^{\frac{3}{2}}$ by $x^{\frac{1}{2}}+y^{\frac{1}{2}}$.

10. Multiply $x^{\frac{7}{2}}-x^3+x^{\frac{5}{2}}-x^2+x^{\frac{3}{2}}-x+x^{\frac{1}{2}}-1$ by $x^{\frac{1}{2}}+1$.

11. Simplify $\left(x^{\frac{3}{2}}\div x^{\frac{4}{7}}\right)^{\frac{8}{13}}$.

12. Simplify $\left(x^{\frac{2}{3}}y^{-1}z^{\frac{1}{4}}\right)^2$.

13. Simplify $\sqrt{x^{\frac{2}{3}}y^{-1}z^{\frac{1}{4}}}$.

14. Find the continued product of

$$x^{\frac{1}{3}},\quad x^{-\frac{3}{4}},\quad (\sqrt[3]{x})^4,\quad x^{\frac{1}{12}},\quad (\sqrt[8]{x})^{\frac{25}{3}},\quad \left(x^{-\frac{7}{4}}\right)^{\frac{7}{6}}.$$

15. Multiply $x^{\frac{3n}{2}}-x^{-\frac{3n}{2}}$ by $\left|x^{\frac{n}{2}}-x^{-\frac{n}{2}}\right|^{-1}$

16. Simplify $\left[\left\{\dfrac{x^{-2r}}{y^{-2n}}\right\}^{-\frac{p}{r}}\right]^{\frac{q}{2n}}$

17. Simplify $\dfrac{(x^{rn})^{-\frac{1}{n}}}{(x^{rn})^{-\frac{1}{r}}}$.

18. Simplify $\dfrac{x^{\frac{3}{2}}-xy^{-\frac{1}{2}}+x^{\frac{1}{2}}y^{-1}-y^{-\frac{3}{2}}}{x^{\frac{5}{2}}-x^2y^{-\frac{1}{2}}+x^{\frac{3}{2}}y^{-1}-xy^{-\frac{3}{2}}+x^{\frac{1}{2}}y^{-2}-y^{-\frac{5}{2}}}$

19. Simplify $\dfrac{\left\{(x^p)^{\frac{1}{r}}(x^q)^{\frac{1}{n}}\right\}^{nr}}{\left\{(\sqrt[q]{y})^n(\sqrt[p]{y})^r\right\}^{pq}} \div \left\{\left|\dfrac{x}{y}\right|^q\right\}^r$

20. Simplify $\dfrac{3(2x)^{\frac{1}{2}}(3bx)^0}{(ax)^{\frac{1}{2}}\sqrt{6a}}$.

CHAPTER III.

RADICAL QUANTITIES AND IRRATIONAL EXPRESSIONS.

1. From the last chapter the student has learned that there are two methods in use for indicating the root of a quantity, one by the ordinary radical sign and the other by a fractional exponent. Of course it is entirely unnecessary to have two modes of writing the same thing, and in this sense either one of the two ways may be considered superfluous. But practically each method of notation has an advantage in special cases, and the student will feel this as he proceeds. This fact that both methods are better than either one, accounts for the retention of both in mathematics.

2. HISTORICAL NOTE—The introduction of the present symbols into algebra was very gradual, and the use of a particular symbol did not generally become common until some time after its suggestion. The signs $+$ and $-$ were first used at the beginning of the 16th century in the works of Grammateus, Rudolf and Stifel. Recarde (born about 1500) is said to have invented the sign of equality about this time. Scheubet's work (1552) is the first one containing the sign $\sqrt{}$, and Vieta (born 1540) first used the vinculum in connection with it. Before this, root-extraction was indicated by a symbol something like ℞. Stevin (born 1548) first used numbers to indicate powers of a quantity, and he even suggested the use of fractional exponents, but not until Descartes (born 1596) did exponents take the modern form of a superior figure.

The development of the general notion of an exponent (negative, fractional, incommensurable) first appears in a work of John Wallis (published in 1665) in connection with the quadrature of plane curves.

To show the appearance of mathematical works before the introduction of the common symbols, we give the following expression taken from Cardan's works (1545):

℞ *v. cu.* ℞ 108 $\bar{p}$. 10 | $\bar{m}$ ℞ *cu.* ℞ 108 $\bar{m}$ 10,

which is an abbreviation for "Radix universalis cubica radicis ex 108 plus 10, minus radice universali cubica radicis ex 108 minus 10." Or, in modern symbols,

$$\sqrt[3]{\sqrt{108}+10}-\sqrt[3]{\sqrt{108}-10}$$

Here is a sentence from Vieta's work (1615).

Et omnibus per E cubum ductis et ex arte concinnatus,

E cubi quad. $+$ Z solido 2 in E cubum, acquabitur B plani cubo.

This translated reads: Multiplying both members ("all") by E^3 and uniting like terms,

$$E^6+2\frac{E}{Z}=B^3$$

3. Definitions. In the following pages, by the word *Radical* may be understood the indicated root of an expression, whether that root is indicated by the ordinary radical sign or by a fractional exponent.

By the *Index* of a radical may be understood either the number written in the angle of the radical sign or the denominator of the fractional exponent.

A multiplier written before a radical will sometimes be called the *co-efficient* of the radical.

A *Simple radical* is the indicated root of a rational expression.

A *Complex radical* is the indicated root of an irrational expression.

A *monomial Surd* is the name applied to the indicated root of a commensurable number, when that root cannot be exactly taken; as $\sqrt{\frac{2}{3}}$, or $\sqrt[4]{3}$.

If all the irrational terms in a binomial or polynomial are surds, it is called a *binomial* or *polynomial surd*, as the case may be.

It should be noticed here that we make a distinction between the terms irrational expression and surd, a distinction which is not commonly made, the two terms being generally defined as identical. According to the above definition, $\sqrt{4}$, $\sqrt[3]{2+\sqrt{2}}$, $\sqrt{\sqrt[3]{3}}$, $\sqrt{\pi}$ are not surds. But they are irrational by the definition of I, Art. 3. This limited meaning of the word surd is convenient and is growing in use. It is found in both Aldis' and Chrystal's algebras.

Radicals are said to be Similar when they have the same index and the expressions under the radical signs are the same; that is, two radicals are similar when they differ only in their coefficients. Such are $5\sqrt{ab}$ and $m\sqrt{ab}$; also $\frac{2}{3}\sqrt[3]{\frac{5}{7}}$ and $\frac{3}{4}\sqrt[3]{\frac{5}{7}}$.

4. Definition. For a radical to be in its *simplest form* it is necessary (1) that no factor of the expression under the radical sign is a perfect power of the required root; (2) that the expression under the radical sign is integral; (3) that the index of the radical is the smallest possible.

It will be seen from the following pages that every simple radical can be placed in this form without changing its value. The transpositions necessary to effect the reductions depend upon certain principles, or theorems, established in the last chapter, which we collect here for reference.

5. *The n th root of the product of several quantities is equal to the product of the n th roots of the several quantities.*

That is, $\sqrt[n]{abc\,.\,.} = \sqrt[n]{a}\,\sqrt[n]{b}\,\sqrt[n]{c}\,.\,.$

or $(a\,b\,c\,.\,.)^{\frac{1}{n}} = a^{\frac{1}{n}}\,b^{\frac{1}{n}}\,c^{\frac{1}{n}}\,.\,.$

6. *The n th root of the quotient of two numbers is equal to the quotient of their n th roots.*

That is, $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

or $\left\{\frac{a}{b}\right\}^{\frac{1}{n}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}$

7. *The nr th root of a quantity equals the n th root of the r th root of the quantity.*

That is, $\sqrt[nr]{a} = \sqrt[n]{\sqrt[r]{a}}$,

or $(a)^{\frac{1}{nr}} = \left\{(a)^{\frac{1}{r}}\right\}^{\frac{1}{n}}$

8. TO REMOVE A FACTOR FROM BENEATH THE RADICAL SIGN. When any factor of the quantity beneath the radical sign is an exact power of the indicated root, the root of that factor may be taken and written as a coefficient while the other factors are left beneath the radical sign. Thus $\sqrt{128}$ may be written $\sqrt{64 \times 2}$, which, by Art. 5, equals $\sqrt{64} \times \sqrt{2}$, which equals $8\sqrt{2}$. As another case take $\sqrt[3]{16ax^4}$, which equals $\sqrt[3]{8x^3 \times 2ax} = \sqrt[3]{8x^3} \times \sqrt[3]{2ax}$ $= 2x\sqrt[3]{2ax}$. It is readily seen that this same process may be applied to any similar case.

9. EXAMPLES. Remove as many factors as possible from beneath the radical signs in the following:

1. $\frac{1}{2}\sqrt{50}$.

2. $\sqrt{810}$.

3. $\sqrt[3]{810}$.

4. $\frac{1}{4}\sqrt{72x^3y^2}$.

5. $\sqrt[3]{2808}$.

6. $\sqrt{50a^{50}}$.

7. $\sqrt[3]{81a^2b^4x^6}$

8. $\sqrt[4]{\dfrac{16a^6x^2y^3}{xy}}$.

9. $\dfrac{1}{x^2}\sqrt{ax^2+2x^3}$.

10. $\sqrt{18a^5b^3}$.

11. $\sqrt[6]{192a^7xy^{12}}$.

12. $\sqrt[r]{(a-b)^n(b+a)^{r+3}}$.

10. TO INTEGRALIZE THE EXPRESSION UNDER THE RADICAL SIGN. Suppose we wish to transform the radical

$$\sqrt[3]{\frac{ab^2}{xy^2}}$$

so that there shall be no fraction under the radical sign. Multiply both numerator and denominator of the fraction by a quantity that will render the denominator a perfect cube, thus:

$$\sqrt[3]{\frac{ab^2}{xy^2}}=\sqrt[3]{\frac{ab^2}{xy^2}\times\frac{x^2y}{x^2y}}=\sqrt[3]{\frac{ab^2x^2y}{x^3y^3}}$$

But, by Art. 6,

$$\sqrt[3]{\frac{ab^2x^2y}{x^3y^3}}=\frac{\sqrt[3]{ab^2x^2y}}{\sqrt[3]{x^3y^3}}=\frac{1}{xy}\sqrt[3]{ab^2x^2y}.$$

In general, to integralize a radical of the form $\sqrt[n]{\frac{a}{b}}$, multiply numerator and denominator by b^{n-1} and we obtain

$$\sqrt[n]{\frac{ab^{n-1}}{b^n}},$$

which, by Art. 6, equals

$$\frac{\sqrt[n]{ab^{n-1}}}{\sqrt[n]{b^n}},$$

and this is equal to

$$\frac{1}{b}\sqrt[n]{ab^{n-1}},$$

which is in the required form.

11. EXAMPLES. Integralize the expressions under the radicla signs in the following, simplifying the result in each case by Art. 8, if necessary:

1. $5\sqrt{\frac{2}{147}}$.

Process: $5\sqrt{\frac{2}{147}}=5\sqrt{\frac{2}{49\times 3}}=5\sqrt{\frac{6}{49\times 9}}=\frac{5}{21}\sqrt{6}$.

2. $\sqrt{\frac{27}{50}}$.

3. $\frac{2}{8}\sqrt{\frac{1}{8}}$.

4. $\sqrt[3]{\frac{2}{8}}$.

5. $\sqrt{\frac{a^4c}{x^3}}$.

6. $\sqrt{2\frac{2}{8}}+\sqrt[4]{\frac{1}{8}}$.

7. $\sqrt[3]{\frac{11}{5}}$.

8. $\sqrt[4]{\frac{324}{2804}}$.

9. $\sqrt[8]{\frac{3x+6y}{4x^2-8y^2}}$.

10. $\sqrt[n]{\frac{a^n+b^n}{(a-b)^{n-r}}}$.

12. TO LOWER THE INDEX OF A RADICAL. It is plain that $\sqrt[4]{25}$ by Art. $7=\sqrt{\sqrt{25}}=\sqrt{5}$; also that $\sqrt[4]{4a^2}=\sqrt{\sqrt{4a^2}}=\sqrt{2a}$; similarly $\sqrt[6]{4b^2}=\sqrt[3]{\sqrt{4b^2}}=\sqrt[3]{2b}$; and in general, $\sqrt[nr]{a^n}=\sqrt[r]{\sqrt[n]{a^n}}=\sqrt[r]{a}$. From this we see that the index of a radical can be lowered if the expression under the radical is a perfect power corresponding to some factor of the original index.

13. EXAMPLES. Reduce the following to their simplest forms. See Art. 4.

1. $\sqrt[4]{36x^2y^4}$.

2. $3\sqrt[6]{4a^2}$.

3. $\sqrt[6]{256c^2x^8}$.

4. $\sqrt[6]{\frac{1000}{x^3}}$.

5. $\sqrt[9]{\frac{a^3}{8b^6}}$.

6. $\sqrt[6]{\dfrac{576(x+y)^2}{9x^2-18xy+9y^2}}$.

7. $2\sqrt[3]{a-b}-\sqrt[6]{64b^2+64a^2-128ab}$.

14. To introduce a Coefficient under the Radical Sign. It is sometimes convenient to have a radical in a form without a coefficient. The coefficient can always be introduced under the radical sign by the inverse of the method of Art. 8. Thus, $2x\sqrt[3]{2ax}=\sqrt[3]{8x^3}\sqrt[3]{2ax}=\sqrt[3]{16ax^4}$; similarly, $a\sqrt[n]{c}=\sqrt[n]{a^nc}$.

15. Examples. Place the coefficients in the following under the radical sign without changing the value of the expression:

1. $3ax^2\sqrt{3ax}$.
2. $\frac{1}{8}\sqrt{6}$.
3. $x\sqrt{a-x}$.
4. $50\sqrt{50}$.
5. $a^2b^2\sqrt{x-y}$.

16. Addition and Subtraction of Radicals. Similar radicals (Art. 4) can be combined by addition or subtraction; and if they are dissimilar no combination can take place. Take for example the expression,

$$\sqrt[4]{9a^6x^8}+2\sqrt{\tfrac{1}{10}}-2\sqrt{\frac{a^3x^4}{3}}+\sqrt{10}.$$

Reducing each expression to its simplest form, it becomes

$$ax^2\sqrt{3a}+\tfrac{1}{5}\sqrt{10}-\tfrac{2}{3}ax^2\sqrt{3a}+\sqrt{10}.$$

It is now noticed that the first and third and the second and fourth radicals are similar to each other; whence, grouping similar terms, the expression becomes

$$(ax^2-\tfrac{2}{3}ax^2)\sqrt{3a}+(1+\tfrac{1}{5})\sqrt{10},$$

$$\text{or } \tfrac{1}{3}ax^2\sqrt{3a}+1\tfrac{1}{5}\sqrt{10}.$$

We observe here the necessity of reducing each of the radicals in any given expression to its simplest form, for then it can be told whether or not any number of the radicals are similar to each other and consequently whether or not they can be combined together.

17. EXAMPLES. Give the value of each of the following expressions in as simple a form as possible :

1. $10\sqrt{\frac{4}{5}}+\sqrt[6]{1000}$.

2. $\frac{2}{3}\sqrt{\frac{3}{2}}-\frac{3}{2}\sqrt{\frac{2}{3}}$.

3. $2\sqrt{48}+3\sqrt{147}-5\sqrt{75}$.

4. $\sqrt{98}+\sqrt{72}+\sqrt{242}$.

5. $\sqrt{75a^4}+\sqrt{27b^5}+\sqrt{48a^2b^2}$.

6. $\frac{1}{2}\sqrt[3]{40x^2}-3\sqrt[3]{625x^6}+10\sqrt[3]{5000x^2}$.

7. $\frac{3x^2}{7y^2}\sqrt{\frac{49y^3}{18x^3}}-\frac{x}{y}\sqrt{\frac{2y^3}{x}}+\frac{3y^2}{5x^3}\sqrt{\frac{50x^5}{81y}}$.

8. $\frac{2}{5}\sqrt{\frac{3}{5}}+\frac{7}{8}\sqrt{\frac{20}{147}}-\frac{1}{2}\sqrt{\frac{5}{3}}$.

9. $\sqrt[n]{\frac{b}{x^n}}-\sqrt[n]{2^n b}+\sqrt[2n]{x^{2n}b^2}$.

10. Prove $\sqrt{\frac{a^2x+x^3-2ax^2}{a^2+2ax+x^2}}+\sqrt{\frac{a^2x+x^3+2ax^2}{a^2-2ax+x^2}}=2\left\{\frac{a^2+x^2}{a^2-x^2}\right\}\sqrt{x}$

18. MULTIPLICATION AND DIVISION OF RADICALS. The product of several radicals of the same index may be expressed as a single radical by means of Art. 5. Thus

$$\sqrt{2}\times\sqrt{3}\times\sqrt{5}=\sqrt{2\times3\times5}=\sqrt{30};$$
$$\sqrt[3]{r^2x^2}\times\sqrt[3]{mx}\times\sqrt[3]{r^2m}=\sqrt[3]{x^3r^4m^2}=rx\sqrt[3]{rm^2};$$
$$\sqrt[n]{a}\times\sqrt[n]{b}\times\sqrt[n]{c}\ .\ .\ .=\sqrt[n]{a\ b\ c\ .\ .\ .}$$

The result should always be reduced to its simplest form. If there are coefficients they should be multiplied together for a new coefficient, for

$$a\sqrt[n]{x}\ b\sqrt[n]{y}\ c\sqrt[n]{z}=abc\sqrt[n]{x}\ \sqrt[n]{y}\ \sqrt[n]{z}=abc\sqrt[n]{xyz}.$$

The quotient of one radical by another of the same index may be expressed as a single radical by means of Art. 6. Thus

$$\sqrt{5}\div\sqrt{7}=\frac{\sqrt{5}}{\sqrt{7}}=\sqrt{\frac{5}{7}}=\frac{1}{7}\sqrt{35}$$
$$\sqrt[n]{a}\div\sqrt[n]{b}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\frac{a}{b}}=\frac{1}{b}\sqrt[n]{ab^{n-1}}$$

The result should always be expressed in its simplest form.

If we wish to multiply or divide radicals of different indices we must first reduce them to a common index. This can be done by

expressing the radical by means of fractional exponents and then reducing them to a common denominator, Thus

$$\sqrt[3]{4}\times\sqrt[4]{3}\times\sqrt{2}=4^{\frac{1}{3}}3^{\frac{1}{4}}2^{\frac{1}{2}}$$

$$=4^{\frac{4}{12}}\,3^{\frac{3}{12}}\,2^{\frac{6}{12}}\text{ (by Art. 8)}=\sqrt[12]{4^4\times3^3\times2^6}=2\sqrt[12]{108}.$$

Also
$$\frac{\sqrt[5]{5}}{\sqrt[3]{3}}=\frac{5^{\frac{3}{15}}}{3^{\frac{5}{15}}}=\sqrt[15]{\frac{5^3}{3^5}}=\tfrac{1}{3}\sqrt[15]{5^3 3^{10}}.$$

19. Examples. Find the value of each of the following expressions:

1. $\sqrt{3ac}\times\sqrt{2am}\times\sqrt{6ax}$.

2. $\sqrt{\frac{2}{3}}\times\sqrt{\frac{3}{4}}\times\sqrt{\frac{1}{2}}$.

3. $\sqrt{\frac{1}{2}}\times\sqrt[3]{\frac{1}{4}}$.

4. $a^2\sqrt{a-x}\times x^2\sqrt{a+x}$.

5. $\sqrt[3]{\frac{3}{10}}\times\sqrt[3]{\frac{5}{12}}$.

6. $2x^{\frac{1}{3}}\times 3x^{\frac{1}{5}}$.

7. $b^r\sqrt[n]{\frac{1}{a}}\times a^n\sqrt[r]{\frac{1}{b}}$.

8. $\sqrt[3]{24}\times6\sqrt[6]{3}$.

9. $\sqrt[3]{9}\div\sqrt[3]{15}$.

10. $2\sqrt[5]{6}\div6\sqrt{2}$.

11. $\sqrt{\frac{1}{2}}\div\sqrt[3]{\frac{1}{2}}$.

12. $\sqrt{2}\div\sqrt[5]{2}$.

13. $\sqrt{a^2-x^2}\div\sqrt{a-x}$.

14. $2\sqrt[5]{\frac{a}{x}}\div3\sqrt[5]{\frac{x}{a}}$.

15. $5\div\sqrt{3}$.

16. $\frac{a+x}{\sqrt{a-x}}\div\frac{a-x}{\sqrt{a+x}}$.

A polynomial involving radicals is generally more easily multiplied or divided by another such polynomial by first expressing the radicals by fractional exponents. As shown in Chapter II, the work will then be no different in principle from the case when the exponents are integral. But in a few of the simpler instances it is unnecessary to pass to fractional exponents, *e. g.*

17. Multiply $3-\sqrt{6}$ by $\sqrt{2}-\sqrt{3}$.

Process:

$$\begin{array}{r} 3-\sqrt{6} \\ \sqrt{2}-\sqrt{3} \\ \hline 3\sqrt{2}-2\sqrt{3} \\ -3\sqrt{3}+3\sqrt{2} \\ \hline 6\sqrt{2}-5\sqrt{3} \end{array}$$ *Result.*

Find the value of

18. $(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})$.

19. $(2\sqrt{5}-3\sqrt{2}+\sqrt{10})(\sqrt{15}-\sqrt{6})$.

20. $(\sqrt[3]{4}-\sqrt{6})(3\sqrt[3]{6}+4\sqrt{3})$.

21. $(3\sqrt{45}-7\sqrt{5})(\sqrt{1\frac{4}{5}}+2\sqrt{9\frac{4}{5}})$.

20. POWERS AND ROOTS OF RADICALS. It has already been shown (Chapter II) that we can raise a quantity affected by a fractional exponent to any required power by multiplying the fractional exponent by the exponent of the required power. It was also shown that any root of a quantity affected with a fractional exponent could be found by dividing the fractional exponent by the index of the required root. Hence we can find any power or any root of a radical if it is expressed by means of fractional exponents; but of course in the simpler cases the convenience of fractional exponents will not be felt. We give one or two illustrations and leave the student to his own method; the chief requirement is that he should be able to show that his work is established on sound principles.

The result should appear in its simplest form.

21. EXAMPLES.

1. Square $\frac{2}{3}\sqrt[3]{a^2}$.

Process: $(\frac{2}{3}\sqrt[3]{a^2})^2=\frac{4}{9}(\sqrt[3]{a^2})^2=\frac{4}{9}\sqrt[3]{a^4}=\frac{4}{9}a\sqrt[3]{a}$.

2. Find the fourth power of $\frac{1}{3}\sqrt[3]{12}$.

3. Cube $\sqrt{\frac{x}{a}}$.

4. Square $3\sqrt[6]{3}$.

5. Square $\frac{1}{2}\sqrt[6]{2}$.

6. Cube $ax\sqrt{ax}$.

7. Find the value of $\left\{\sqrt[n]{\frac{a^n-b^n}{x^{n-r}}}\right\}^{r-n}$

8. Find the value of $\left\{\frac{\sqrt{a^2-x^2}}{\sqrt[3]{a+x}}\right\}^2$

9. Find the cube root of $\frac{8}{9}\sqrt{3}$.

One process: $\sqrt[3]{\frac{8}{9}\sqrt{3}}=\sqrt[3]{\frac{8}{9}}\sqrt[3]{\sqrt{3}}$ (Art. 5) $=\sqrt[3]{\frac{8}{9}}\sqrt[6]{3}$, (Art. 7)

$=\sqrt[6]{\frac{64}{81}\times 3}$ (Art. 18) $=2\sqrt[6]{\frac{1}{27}}=2\sqrt{\frac{1}{3}}=\frac{2}{3}\sqrt{3}$.

Another process; $\frac{8}{9}\sqrt{3}=\sqrt{\frac{64}{27}}$, (Art. 13)

$$\sqrt[3]{\sqrt{\frac{64}{27}}}=\sqrt[6]{\frac{64}{27}}=\sqrt{\frac{4}{3}}=2\sqrt{\frac{1}{3}}=\frac{2}{3}\sqrt{3}.$$

10. Find the fourth root of $\frac{1}{16}x^2\sqrt[3]{9x^2y^2}$.

11. Take the cube root of $3\sqrt{3}$.

12. Take the cube root of $\sqrt{\frac{1}{5}}$.

13. Find the value of $\sqrt[3]{\frac{c}{y}\sqrt{\frac{c}{y}}}$

14. Simplify $\sqrt{\frac{9x^{\frac{8}{5}}}{98y^{\frac{3}{2}}}}$

15. Find the value of $(\frac{3}{2}\sqrt[3]{\frac{4}{9}})^2\times\sqrt[3]{\frac{2}{3}\sqrt{\frac{81}{8}}}$.

16. Simplify $\sqrt[3]{\frac{\sqrt{b^2-y^2}}{b+y}}$

22. RATIONALIZING FACTORS. Any multiplier which, when applied to an irrational polynomial, will free it from radicals is called a Rationalizing Factor. Thus $5-\sqrt{3}$ is a rationalizing factor for the binomial surd $5+\sqrt{3}$, since $(5+\sqrt{3})(5-\sqrt{3})=25-3=22$. Similarly the rationalizing factor for $\sqrt{3}-\sqrt{5}$ is $\sqrt{3}+\sqrt{5}$, since $(\sqrt{3}-\sqrt{5})(\sqrt{3}+\sqrt{5})=5-3=2$.

For larger polynomials it may be that the ratfonalizing factor is itself composed of several factors. Take the quadratic trinomiai surd $\sqrt{2}-\sqrt{3}$. The rationalizing factor is

$(\sqrt{2}-\sqrt{3}+\sqrt{7})(2-2\sqrt{6}$, for

$$(\sqrt{2}-\sqrt{3}+\sqrt{7})(\sqrt{2}-\sqrt{3}-\sqrt{7})(2-2\sqrt{6})$$
$$=[(\sqrt{2}-\sqrt{3})^2-(\sqrt{7})^2](2-2\sqrt{6})$$
$$=-(2+2\sqrt{6})(2-2\sqrt{6})=20.$$

23. PROBLEM. TO RATIONALIZE ANY BINOMIAL QUADRATIC SURD. Any binomial quadratic surd may be represented by $a\sqrt{p}+b\sqrt{q}$, where a and b may be either positive or negative. The rationalizing factor is plainly $a\sqrt{p}-b\sqrt{q}$, for

$$(a\sqrt{p}+b\sqrt{q})(a\sqrt{p}-b\sqrt{q})=a^2p-b^2q,$$

which is rational.

24. PROBLEM. TO RATIONALIZE ANY TRINOMIAL QUADRATIC SURD. Any trinomial quadratic surd may be represented by $a\sqrt{p}+b\sqrt{q}+c\sqrt{r}$, where a, b, and c are supposed to be any rational quantities whatever, positive or negative or integral or fractional. Multiply first by $a\sqrt{p}+b\sqrt{q}-c\sqrt{r}$, and we obtain

$$(a\sqrt{p}+b\sqrt{q}+c\sqrt{r})(a\sqrt{p}+b\sqrt{q}-c\sqrt{r})=(a\sqrt{p}+b\sqrt{q})^2-(c\sqrt{r})^2$$
$$=a^2p+b^2q-c^2r+2ab\sqrt{pq}, \qquad (1)$$

which is rational as far as r is concerned. Now multiply this by

$$(a^2p+b^2q-c^2r)-2ab\sqrt{pq} \qquad (2)$$

and we obtain

$$\{(a^2p+b^2q-c^2r)+2ab\sqrt{pq}\}\{(a^2p+b^2q-c^2r)-2ab\sqrt{pq}\}$$
$$=(a^2p+b^2q-c^2r^2)^2-4a^2b^2pq, \qquad (3)$$

which is rational with respect to all the quantities. The rationalizing factor for the original trinomial quadratic surd is thus seen to be

$$(a\sqrt{p}+b\sqrt{q}-c\sqrt{r})(a^2p+b^2q-c^2r-2ab\sqrt{pb}). \qquad (4)$$

25. The second parenthesis in (4) above will be found to be composed of the two factors

$$(a\sqrt{p}-b\sqrt{q}+c\sqrt{r})(a\sqrt{p}-b\sqrt{q}-c\sqrt{r})$$

Hence the rationalizing factor of $a\sqrt{p}+b\sqrt{g}+c\sqrt{r}$ may be written

$$(a\sqrt{p}+b\sqrt{q}-c\sqrt{r})(a\sqrt{p}-b\sqrt{q}+b\sqrt{r})(a\sqrt{p}-b\sqrt{q}-c\sqrt{r})$$

Observe that the terms of each of the component trinomial factors of this expression are those of the given irrational quantity and the signs are those exhibited in the scheme—

$$\begin{matrix} + & + & - \\ + & - & + \\ + & - & - \end{matrix}$$

Now it is evident that, keeping the first sign unchanged, there is no other arrangement of signs than those written in this scheme, except the arrangement $+ + +$, which is the arrangement of the given trinomial. Therefore

The rationalizing factor for any trinominal quadratic surd is the product of all the different trinomials which can be made from the original by keeping the first term unchanged and giving the signs $+$ and $-$ to all the remaining terms in every possible order, except the order occurring in the given trinomial.

As an example, find the rationalizing factor for $\sqrt{5}-\sqrt{7}+\sqrt{3}$. The above method shows it to be

$$(\sqrt{5}-\sqrt{7}-\sqrt{3})(\sqrt{5}+\sqrt{7}+\sqrt{3})(\sqrt{5}+\sqrt{7}-\sqrt{3})$$

and multiplying the original trinomial by this the rationalized result is found to be -40.

The above problem is cabable of generalization, but its proof cannot be practically given here. The generalized statement is as follows:

The rationalizing factor for any polynomial quadratic surd is the product of all the different polynomials which can be made from the original by keeping the first term unchanged and giving the signs $+$ and $-$ to all the remaining terms in every possible order except the order occurring in the given polynomial.

26. PROBLEM. TO RATIONALIZE ANY BINOMIAL SURD. A binomonial surd will either take the form $a^{\frac{b}{r}}+c^{\frac{p}{q}}$ or $a^{\frac{b}{r}}-c^{\frac{p}{q}}$. Now since these fractional exponents may be reduced to a common denominator so that the expressions become $a^{\frac{bq}{rq}}+c^{\frac{pr}{rq}}$ or $a^{\frac{bq}{rq}}-c^{\frac{pr}{rq}}$ these binominal surds may be supposed in the form $a^{\frac{s}{n}}+c^{\frac{t}{n}}$ or $a^{\frac{s}{n}}-c^{\frac{t}{n}}$. These, then, are the only forms necessary to consider, since all binomial surds are reducible thereto.

(a) To rationalize the form $a^{\frac{s}{n}}-c^{\frac{t}{n}}$.

For convenience let $a^{\frac{s}{n}}=x$ and $c^{\frac{t}{n}}=y$; when $a^{\frac{s}{n}}-c^{\frac{t}{n}}=x-y$. Now multiply $x-y$ by

$$x^{n-1}+x^{n-2}y+x^{n-3}y^2+ \; . \; . \; . \; y^{n-1} \qquad (1)$$

and we obtain

$$(x-y)(x^{n-1}+x^{n-2}y+x^{n-3}y^2+\ .\ .\ .\ y^{n-1})=x^n-y^n,$$

by substitution, I, Art. 26. But

$$x^n-y^n=\left[a^{\frac{s}{n}}\right]^n-\left[c^{\frac{t}{n}}\right]^n=a^s-c^t,$$

which is rational. Therefore (1) is the rationalizing factor for $a^{\frac{s}{n}}-c^{\frac{t}{n}}$.

(b) To rationalize the form $a^{\frac{s}{n}}+c^{\frac{t}{n}}$.

As before, let $a^{\frac{s}{n}}=x$ and $c^{\frac{t}{n}}=y$; whence $a^{\frac{s}{n}}+c^{\frac{t}{n}}=x+y$. Multiply $x+y$ by

$$x^{n-1}-x^{n-2}y+x^{n-3}y^2-\ .\ .\ .\ \pm y^{n-1}. \qquad (2)$$

The product is

$$(x+y)(x^{n-1}-x^{n-2}y+x^{n-3}y^2-\ .\ .\ .\ \pm y^{n-1})=\begin{cases} x^n-y^n \text{ if } n \text{ is even,} \\ x^n+y^n \text{ if } n \text{ is odd,}\end{cases}$$

by I, Arts. 27, 28. But

$$x^n-y^n=\left[a^{\frac{s}{n}}\right]^n-\left[c^{\frac{t}{n}}\right]^n=a^s-c^t,$$

$$x^n+y^n=\left[a^{\frac{s}{n}}\right]^n+\left[c^{\frac{t}{n}}\right]^n=a^s+c^t.$$

Both of these results are rational; therefore (2) is the rationalizing factor.

27. EXAMPLES.

1. Rationalize $d^{\frac{2}{3}}-r^{\frac{1}{2}}$.

With a common denomininator for the exponents this becomes $d^{\frac{4}{6}}-r^{\frac{3}{6}}$; whence $n=6$, $s=4$, $t=3$; then $x=d^{\frac{4}{6}}$, $y=r^{\frac{3}{6}}$.

$$(x-y)(x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5)$$

$$=\left[d^{\frac{4}{6}}-r^{\frac{3}{6}}\right]\left[d^{\frac{20}{6}}+d^{\frac{16}{6}}r^{\frac{3}{6}}+d^{\frac{12}{6}}r^{\frac{6}{6}}+d^{\frac{8}{6}}r^{\frac{9}{6}}+d^{\frac{4}{6}}r^{\frac{12}{6}}+r^{\frac{15}{6}}\right]$$

$$=\left[d^{\frac{2}{3}}-r^{\frac{1}{2}}\right]\left[d^{\frac{10}{3}}+d^{\frac{8}{3}}r^{\frac{1}{2}}+d^{\frac{6}{3}}r^{\frac{2}{2}}+d^{\frac{4}{3}}r^{\frac{3}{2}}+d^{\frac{2}{3}}r^{\frac{4}{2}}+r^{\frac{5}{8}}\right]$$

$$=d^4-r^3, \text{ which is rational.}$$

2. Rationalize $6+3\sqrt[4]{5}$.

With a common denominator for the fractional exponents this becomes $6^{\frac{4}{4}}+(3^4\times5)^{\frac{1}{4}}$; whence $n=4$, $s=4$, $t=1$; then $x=6^{\frac{4}{4}}$ and $y=(3^4\times5)^{\frac{1}{4}}$. Therefore

$$(x+y)(x^3-x^2y+xy^2-y^3)$$

$$=\left\{6^{\frac{4}{4}}+(3^4\times5)^{\frac{1}{4}}\right\}\left\{6^3-6^2\times3\times5^{\frac{1}{4}}+6\times3^2\times5^{\frac{1}{2}}-3^3\times5^{\frac{3}{4}}\right\}$$
$$=6^4-3^4\times5=891.$$

3. Rationalize $\sqrt[3]{2}+2\sqrt[3]{9}$.

28. RATIONALIZATION OF THE DENOMINATORS OF FRACTIONS. The most common application of rationalizing factors is in the rationalization of the denominators of irrational fractions. Considerable labor is saved in computing the value of a numerical irrational fraction if we first rationalize the denominator. Thus, to compute the value of $\frac{\sqrt{5}}{\sqrt{7}-\sqrt{2}}$ correct to five decimal places, three square roots must be taken and one of them must be divided by the difference of the other two. Now, it will be obvious on reflection that these square roots must be taken to nearly ten places of decimals if we are to be *absolutely certain* that five decimal places of the quotient are correct. It will be easily seen how much more readily the value can be found after the denominator has been rationalized. Multiplying both numerator and denominator by the rationalizing factor for the denominator, we have

$$\frac{\sqrt{5}}{\sqrt{7}-\sqrt{2}}=\frac{\sqrt{5}(\sqrt{7}+\sqrt{2})}{(\sqrt{7}-\sqrt{2})(\sqrt{7}+\sqrt{2})}=\frac{\sqrt{35}+\sqrt{10}}{5}$$

Now but *two* square roots need be taken, and these to no more than five decimal places, since the exact value of the denominator is known.

29. EXAMPLES.

1. Rationalize the denominator of $\frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}}$.

2. Rationalize the denominator of $\frac{5}{\sqrt{2}+\sqrt[3]{3}}$.

3. Prove $\left\{\frac{2}{2+\sqrt{2}}\right\}^{\frac{1}{2}}=(2-\sqrt{2})^{\frac{1}{2}}$.

4. Given $\sqrt{3}=1.7320508$, find the value of $\frac{1}{2+\sqrt{3}}$.

5. Prove $\frac{(2+\sqrt{3})(3+\sqrt{5})(\sqrt{3}-2)}{(5-\sqrt{5})(1+\sqrt{3})}=\frac{1}{5}\sqrt{15}$.

6. Rationalize the denominator of $\dfrac{4+\sqrt{21}}{\sqrt{7}-\sqrt{3}-\sqrt{2}}$.

7. Rationalize the denominator of $\dfrac{\sqrt{18}}{2\sqrt{2}-\sqrt{3}}$.

8. Rationalize the denominator of $\dfrac{1}{3+2\sqrt{2}}$.

9. What relation must hold between a and x in order that $\dfrac{1}{a+\sqrt{x}}=a-\sqrt{x}$?

10. Rationalize the denominator of $\dfrac{\sqrt{2}-\sqrt{3}+\sqrt{5}}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$.

11. Compute the value of the following to three places of decimals, having first reduced it to its simplest form:

$$\frac{\sqrt{3+\sqrt{5}}-\sqrt{5-\sqrt{5}}}{\sqrt{3+\sqrt{5}}+\sqrt{5-\sqrt{5}}}.$$

12. Prove $\dfrac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}}+\dfrac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}}=2x^2$.

30. THEOREM. *If a, b, p, q are commensurable and $\sqrt{b}$ and $\sqrt{q}$ incommensurable, and if $a+\sqrt{b}=p+\sqrt{q}$, then $a=p$ and $\sqrt{b}=\sqrt{q}$.*

If a does not equal p, suppose $a=p+d$. Substitute this value for a in the given equation, and we have

$$p+d+\sqrt{b}=p+\sqrt{q}$$

or

$$d+\sqrt{b}=\sqrt{q}$$

squaring both sides

$$d^2+2d\sqrt{b}+b=q$$

whence

$$\sqrt{b}=\frac{q-b-d^2}{2d}$$

That is, an incommensurable quantity equals a commensurable, which is absurd. Therefore a cannot differ from p. And if $a=p$, $\sqrt{b}$ must equal $\sqrt{q}$.

A—6

31. EXAMPLES. We append a few miscellaneous examples on the last two chapters.

1. Does $(a+x)^{\frac{1}{2}}=a^{\frac{1}{2}}+x^{\frac{1}{2}}$?

2. Multiply together $\sqrt{a}$, $\sqrt[3]{b^3}$, $\sqrt[3]{c^2}$, $\sqrt[4]{3a^5}$ and $a^{-\frac{7}{2}}$.

3. Simplify $\dfrac{x^{\frac{2}{n}}-y^{\frac{2}{r}}}{x^3-y^3}\times\dfrac{y^6-x^6}{y^{\frac{1}{r}}+x^{\frac{1}{n}}}$.

4. Multiply together

$\sqrt{x^{2n}+x^n y^n+y^{2n}}$, $\sqrt{x^n-y^n}$, $\sqrt{x^{2n}-x^n y^n+y^{2n}}$ and $\sqrt{x^n+y^n}$.

5. Multiply together

$(a^2+ab+b^2)^{\frac{1}{n}}$, $(a-b)^{\frac{1}{n}}$, $(a-b)^{\frac{1}{r}}$ and $(a^2+ab+b^2)^{\frac{1}{r}}$

6. Simplify $$\dfrac{1-\dfrac{a^2-x^2}{a-x}}{\dfrac{\sqrt{a^2-x^2}}{\sqrt{a-x}}+1}$$

7. Simplify $\dfrac{x+\sqrt{x^2-y^2}}{x-\sqrt{x^2-y^2}}-2\dfrac{x^2}{y^2}+1$

8. Cube the expression $a^2\sqrt{x}-\sqrt{ba}\sqrt[5]{y}$.

9. Prove $2+\sqrt{3}$ is the reciprocal of $2-\sqrt{3}$; and find what must be the relation between the two terms x and $\sqrt{y}$ so that $x+\sqrt{y}$ shall be the reciprocal of $x-\sqrt{y}$.

10. Simplyfy $\left\{\dfrac{a^2x}{(x+a)^{\frac{5}{4}}}\right\}^{-\frac{1}{5}}$

11. Simplify the expression $\dfrac{(1+x)^{\frac{1}{2}}+(1-x)^{\frac{1}{2}}}{(1+x)^{\frac{1}{2}}-(1-x)^{\frac{1}{2}}}$

first by rationalizing the numerator, and then by rationalizing the denominator.

12. Prove that if $p=3$ and $q=5$,

$$\frac{e^{p-q}+pq^{-1}+qp^{-1}+e^{q-p}}{pq^{-1}e^{p-q}+2+qp^{-1}e^{q-p}}=\frac{3e^2+5}{3+5e^2}$$

CHAPTER IV.

QUADRATIC EQUATIONS CONTAINING ONE UNKNOWN QUANTITY.

1. DEFINITION. An *Equation of the Second Degree*, or a *Quadratic Equation*, is one where the highest degree of any term with reference to the unknown quantities is two.

It must be remembered that the degree of an equation with reference to any quantity is not spoken of unless the equation is rational and integral with reference to that quantity. See I, Art. 6.

2. We will consider in this chapter quadratic equations containing but one unknown quantity, such as—

$$3x^3+5x=24, \qquad (1)$$

$$2x^2-\tfrac{2}{8}x=.346, \qquad (2)$$

$$\sqrt{3}\,x^2-(\tfrac{3}{5}-\sqrt[3]{7})x=4-\sqrt{5}, \qquad (3)$$

$$\left(m+\frac{r}{s}\right)x^2+(d-t)x=p+\sqrt{k}. \qquad (4)$$

These equations are all obviously quadratics. But some equations, which are irrational or fractional with reference to x in their present form, drop into the quadratic type as soon as the proper transformations are performed. Thus the equation,

$$\sqrt{x}+\frac{\sqrt{b}-\sqrt{a}}{\sqrt{b}}=\frac{1}{\sqrt{x}}+\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}} \qquad (5,\ a)$$

may be made integral with reference to x by multiplying through by $\sqrt{x}$, the resulting form being

$$x+\frac{(\sqrt{b}-\sqrt{a})\sqrt{x}}{\sqrt{b}}=1+\frac{(\sqrt{a}-\sqrt{b})\sqrt{x}}{\sqrt{a}}.$$

Transposing and uniting terms,

$$x+\frac{(b-a)\sqrt{x}}{\sqrt{ab}}=1.$$

Transposing the rational parts to the right hand side of the equation, we obtain the form

$$\frac{(b-a)\sqrt{x}}{\sqrt{ab}}=1-x.$$

Now rationalizing with respect to x, by squaring both sides of the equation, it becomes

$$\frac{(b-a)^2 x}{ab} = 1 - 2x + x^2.$$

Finally transposing and collecting terms, we have

$$x^2 + \frac{a^2+b^2}{ab} x = 1, \qquad (5, b)$$

which is a quadratic equation. While the equation $(5, a)$ has been reduced to the quadratic form $(5, b)$ by apparently legitimate processes, yet we will find that the integralization and rationalization of an equation with reference to the unknown quantity has in general an effect on the solution of the equation *which it is necessary to take into account*, and which renders it possible that the values of x which satisfy $(5, b)$ may not be identical with those that satisfy $(5, a)$. For this reason the treatment of those equations which require the operation of integralization or of rationalization before they are in the quadratic form, is reserved for Chapter VI.

3. TYPICAL FORMS OF THE QUADRATIC. It is evident that equations (1), (2), (3), (4) and $(5, b)$, or any other quadratic equations which can be imagined, may all be said to be of the typical form,

$$ax^2 + bx = c, \qquad (6)$$

where a, b and c are supposed to stand for any numbers whatever, either integral or fractional, positive or negative, or commensurable or incommensurable. Hence $ax^2+bx=c$ is said to be a *typical form* of the quadratic equation.

If we suppose the quadratic equation to be divided through by the coefficient of x^2 the result will be of the form

$$x^2 + px = q, \qquad (7)$$

where p and q are supposed to be any algebraic quantities whatever, fractional or integral, positive or negative, commensurable or incommensurable. This is the second typical form of the quadratic equation, and one which is much used.

4. DEFINITION. A *Root* of an equation is any value of the unknown quantity which satisfies the equation.

Thus 3 is a root of the equation $3x-6=0$, for when substituted

for x it satisfies the equation. Also, both 2 and 3 are roots of the equation $x^2-5x+10=4$, for either of these values when substituted for x will satisfy the equation.

The student must carefully note that this is an entirely different use of the word *root* from that occuring in the expressions *square root*, *cube root*, *etc.*

5. Equations of the second degree are often divided into the two classes of *complete* and *incomplete* quadratics. A complete quadratic is one which contains both the first and second powers of the unknown, as $x^2+px=q$. An incomplete quadratic has the first power of the unknown quantity lacking, and hence can always be placed in the form $x^2=q$, where q is any algebraic quantity conceivable. By some the adjectives *affected* and *pure* are used in place of the words *complete* and *incomplete* respectively.

6. PROBLEM. TO SOLVE ANY INCOMPLETE QUADRATIC.

First, reduce to the form

$$x^2=q$$

by putting all the known quantities on the right hand side of the equation and all the terms containing x^2 on the left hand side, then dividing through by the coefficient of x^2.

Then take the square root of both sides of the equation, remembering that every quantity has two square roots, and we obtain

$$x=\pm\sqrt{q}$$

and the equation is solved.

It might be thought that in taking the square root of both sides of $x^2=q$ we should write

$$\pm x=\pm\sqrt{q}.$$

But, by taking the signs in all possible ways, this gives

$$+x=+\sqrt{q}$$
$$-x=-\sqrt{q}$$
$$+x=-\sqrt{q}$$
$$-x=+\sqrt{q}.$$

Each of the first two of these is equivalent to $x=\sqrt{q}$, and each of the last two is the same as $x=-\sqrt{q}$, and, on the whole, we merely have

$$x=\pm\sqrt{q}.$$

Whence it is seen to be sufficient to write the sign $\pm$ on but one side of the equation.

7. Examples of Incomplete Quadratics.

Solve the following equations:

1. $(3x-\frac{1}{2})(3x+\frac{1}{2})=3\frac{3}{4}$.
2. $(ax-b)(ax+b)=c$.
3. $(x+2)^2+(x-2)^2=24$.
4. $(x+b)^2+(x-b)^2=c$.
5. $(ax+b)^2+(ax-b)^2=c$.
6. $(x+7)(x-9)+(x-7)(x+9)=76$.
7. $(x+a)(x-b)+(x-a)(x+b)=c$.
8. $(x+a)^2=q$.

Show that examples 1, 3 and 6 may be solved by proper substitution in the results to examples 2, 5 and 7 respectively.

8. We have solved the equation $x^2=q$, and also the equation $(x+a)^2=q$ (Ex. 8) in a similar manner. Now it is evident that the equation

$$x^2+px=q$$

can be solved if it can be put in either of the above forms. It can be placed in the form $(x+a)^2=q$ if the first member can be made the square of a binomial. On inspection it is seen that x^2+px are the first two terms of the square of a binomial, the third term of which must be $\frac{1}{4}p^2$. Hence, if we add $\frac{1}{4}p^2$ to both sides of the equation

$$x^2+px=q,$$

it takes the form

$$x^2+px+\frac{1}{4}p^2=q+\frac{1}{4}p^2,$$

or

$$(x+\frac{1}{2}p)^2=q+\frac{1}{4}p^2,$$

which is of the form $\qquad (x+a)^2=q.$

The process of putting a quadratic equation in this form is called *completing the square*.

9. PROBLEM. TO SOLVE THE TYPICAL QUADRATIC $x^2+px=q$.

Add $\frac{1}{4}p^2$ to both members and we obtain

$$x^2+px+\tfrac{1}{4}p^2=q+\tfrac{1}{4}p^2.$$

The left hand member is seen on inspection to be the square of the binomial $(x+\frac{1}{2}p)$; whence taking the square root of both members,

$$x+\tfrac{1}{2}p=\pm\sqrt{q+\tfrac{1}{4}p^2}.$$

Solving this simple equation for x we have

$$x=-\tfrac{1}{2}p\pm\sqrt{q+\tfrac{1}{4}p^2},$$

which gives the two values of x,

$$-\tfrac{1}{2}p+\sqrt{q+\tfrac{1}{4}p^2} \text{ and } -\tfrac{1}{2}p-\sqrt{q+\tfrac{1}{4}p^2}.$$

Hence, *to solve an equation in the form $x^2+px=q$, add the square of one-half the coefficient of x to each side of the equation. Take the square root of both members, and an equation of the first degree is obtained, from which x can be found in the usual way.*

10. PROBLEM. TO SOLVE THE TYPICAL QUADRATIC $ax^2+bx=c$.

Multiply through by $4a$ and obtain

$$4a^2x^2+4abx=4ac.$$

Adding b^2 to both members it becomes

$$4a^2x^2+4abx+b^2=4ac+b^2.$$

The left hand member is seen on inspection to be the square of a binomial; whence, taking the square root of both members, we obtain

$$2ax+b=\pm\sqrt{4ac+b^2},$$

whence, solving this simple equation,

$$x=\frac{-b\pm\sqrt{4ac+b^2}}{2a},$$

which gives the two values of x,

$$\frac{-b+\sqrt{4ac+b^2}}{2a} \text{ and } \frac{-b-\sqrt{4ac+b^2}}{2a}.$$

Hence, *to solve an equation in the form $ax^2+bx=c$, multiply through by four times the coefficient of x^2 and add the square of the coefficient of x to each side of the equation. Then take the square root of both members, and an equation of the first degree will be obtained, from which x can be found.*

11. Historical Note. The origin of the solution of the quadratic equation cannot be definitely traced to any one man or any one race. Algebra, as we now have it, has been a slow growth, and, as we pass back in time, it gradually shades off into the arithmetic of antiquity. Diophantus, an Alexandrian Greek of the fourth century, A. D., who wrote a treatise on arithmetic, could undoubtedly solve quadratic equations, although he devotes no special book to their treatment. But algebra, in a more perfect form, may be traced to the Hindoos. Aryabhata (475 A. D.) was familiar with a solution of the complete quadratic, and Bramagupta (598 A. D.) gives a comparatively elaborate treatment of it. The solution of the quadratic was also known to the Arabs, and a solution with geometric treatment is given by Mohammed ben Musa, of the ninth century. All the early methods of solution consist in what is commonly known as "completing the square" and were substantially the same as those now used.

12. Examples of Complete Quadratic Equations. If a quadratic equation cannot be placed in the form $x^2+px=q$ *without the introduction of fractions*, it is generally advisable before solution to clear it of fractions thereby putting it in the form $ax^2+bx=c$, in which case a, b, and c will be integral. The equation can then be solved by the method of Art. 10, thereby avoiding fractions in the process of solution, which is a great advantage. If the equation takes the form $x^2+px=q$ without p and q being fractional, then a solution by the method of Art. 9 will be better.

To illustrate the common arrangement of the work we solve the following quadratic.

Find the values of x in

The first task is to place the equation in one of the typical forms.

$$3x^2-9x+5=\frac{10-x^2}{3}$$

Clearing of fractions,

$$9x^2-27x+15=10-x^2.$$

Transposing and uniting terms,

$$10x^2-27x=-5.$$

Now "complete the square."

Multiplying through by 4 times 10, and adding $(27)^2$ to both sides, we obtain

Since it has been proved that this method will give a complete square, it is not necessary to work out the value of the coefficient of x, but merely to indicate it by ().

$$400x^2-(\ )x+(27)^2=-200+(27)^2,$$

or $$400x^2-(\ )x+27^2=529.$$

Taking the square root of both members,

$$20x-27=\pm 23,$$

and solving this simple equation

$$20x=50 \text{ or } 4$$

$$x=2\tfrac{1}{2} \text{ or } \tfrac{1}{5}.$$

Solve the following:

1. $x^2+7x+15=5$.

2. $x^2+6x-1=5-20x$.

3. $3x^2+5x=100$.

4. $15x^2-28x+10=5$.

In completing the square in a case like this where the coefficient of x is divisible by 2, fractions can be avoided without multiplying the equation through by 4. Thus:

$$15x^2-28x=-5,$$

multiplying through by 15 (instead of 4×15), and adding the square of one-half of 28, we obtain

$$(15)^2x^2-15\times 28x+(14)^2=-5\times 15+(14)^2$$

which is a complete square.

5. $x^2+12\frac{7}{12}x=38\frac{2}{3}$.

6. $x^2+(x+1)^2=\frac{13}{6}(x+1)$.

7. $x^2+6.51=5.2x$.

8. $x^2+1-ax-\dfrac{p}{q}=0$.

9. $(x-3)(x-5)=0$.

10. $(x-a)(x-b)=0$.

11. $(3x-5)(5x-3)=0$.

12. $(ax-b)(bx-a)=0$.

13. $(x+a-b)(x-a+b)=0$.

14. $a^2-x^2=(a-x)(b+c-x)$.

15. $(33+10x)^2+(56+10x)^2=(65+14x)^2$.

16. $(7-4\sqrt{3})x^2+(2-\sqrt{3})x=2$.

17. $\frac{5}{7}x^2+\frac{7}{5}x+\frac{73}{140}=0$.

18. $x^2+qax=a^2-b^2$.

19. $(x-a)^2=(x-b)(a+b)$.

20. $dez^2-(d^2+e^2)z+de=0$.

21. $x^2-2a(x+b)=2bx-a^2-b^2$.

22. $x^2+10x+30=5$.

23. $a+b+x=a^2b^2x^2$.

24. $\dfrac{x^2}{a}+ax=\dfrac{x^2}{b}+bx$.

25. $ax^2+bx+c=x^2+px+q$.

26. $x^2-1=k(kx^2-4x-k)$.

27. $(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a)=0$.

Result, $x=\frac{1}{3}(a+b+c)\pm\frac{1}{3}\sqrt{a^2+b^2+c^2-ab-bc-ca}$.

28. $x^2+6x+21=10$.

This becomes, in the typical form,

$$x^2+6x=-11.$$

Completing the square, we obtain

$$x^2-6x+9=-2.$$

Now we cannot obtain the square root of the right-hand member of this equation; for it is a *negative quantity*, and the square of no algebraic number can be negative. But, if we were to go through the operation of finding x as has been done in the other cases above, and indicate the root of -2 as if we *could* take it, we would have

$$x=-3\pm\sqrt{-2}.$$

Thus we have had *forced* upon us in the solution of the quadratic equation, something which, whatever interpretation it may have, is evidently *not an algebraic quantity* in the sense in which the term is commonly used. Such an expresion is called an *imaginary*, and its treatment is reserved for Part II of the present work. In the next chapter will be found a discussion of the circumstances under which such expressions occur.

29. $4x^2+4x+4=x^2$.

13. PROBLEMS REQUIRING THE SOLUTION OF QUADRATIC EQUATIONS. The student in his previous study has probably already noticed that the first task in the algebraic solution of a problem is always an attempt to express the language of the problem in algebraic symbols; that is, to cast the relations and conditions expressed by the words of the problem into an equivalent statement in the form of one or more algebraic equations. This work is called the *statement* of the problem, and is generally a difficult one for the beginner to perform. When the statement of a problem is complete, all that remains to be done is the solution of the equation or equations obtained thereby by processes already familiar.

We wish to strongly emphasize the fact that the equation obtained by the translation of the words of most of the algebraic problems in the books is often *not an exact equivalent to the condi-*

tions and relations told in the language of the problem. In fact, the equation often *embraces more* than the problem itself. We will illustrate this by the following problem:

A certain number consists of two digits whose sum is 10. If we reverse the digits and multiply this new number by the original number, the product will be 2944. Required the number.

Statement, or translation of the language into an algebraic equation.

Let x=the digit in unit's place;
then $10-x$=the digit in ten's place,
and $10(10-x)$=the value of the digit in ten's place;
whence $10(10-x)+x$=the value of the original number;
also $10x+(10-x)$=the value of the number with the digits reversed.

But, by the problem,

$$[10(10-x)+x][10x+(10-x)]=2944. \quad (1)$$

Solution of the equation.

That is,

$$(100-9x)(10+9x)=2944.$$

Expanding left member,

$$81x^2-810x-1000=-2944.$$

Transposing and uniting,

$$81x^2-810x=-1944.$$

Dividing through by 81,

$$x^2-10x=-24. \quad (2)$$

Completing square and solving,

$$x=4 \text{ or } 6.$$

The number is therefore either 46 or 64. Now consider equation (1) as a translation of the problem into algebra. As far as is *stated by the equation* (1) the unknown quantity x may be any algebraic quantity conceivable,—positive or negative, integral or fractional, rational or irrational, or, in fact, it may possibly be what we have called an imaginary. As far as the equation expresses the nature of x, it may as likely turn out in the solution one kind as another of those enumerated. But, as *expressed in the language of the problem*, x must be a *digit*; that is, a *positive integral number less than ten*. The equation does not express this fact and cannot be made to do it. The reason why the prob-

lem really works out all right is that it was *made to order*; that is, the number 2944 was especially selected so that the problem would "work out". If we wish this problem stated in words so that it is more nearly identical with its expression in the form of an equation, we must throw out the word "digits" as follows:

There are two numbers whose sum is ten. If ten times the first plus the second is multiplied by ten times the second plus the first, the product will be 2944. Find the numbers.

This is nearly as general as the algebraic equation. It permits of either positive or negative, integral or fractional, commensurable or incommensurable, results, and indeed as the word number is often used it would permit of imaginary results. This problem can be made identical with the original by adding at its close some such caution as this:

Do not obtain a fractional, a negative, nor an incommensurable result, nor any result greater than 9.

It is such conditions as these that we fail to incorporate into an algebraic equation. The algebraic statement, as far as the unknown is concerned, is always the most general possible and contains in it no restriction of the unknown to any particular class of numbers, and for this reason *the algebraic statement of a problem is often more general than the problem itself.* This fact should be remembered, as it will help to explain many apparent difficulties which arise in some problems. These non-algebraic conditions in a problem must be ignored until after the solution is had, and then if a result is obtained like a fractional number of live sheep or a negative price per head, it must be cast out, not because the mathematics is unreliable, but because the problem is cramped and does not fill up the full measure of generality which algebraic methods provide for.

The greatest breadth and elegance of algebraic analysis would be observed in the treatment of problems in geometry, mechanics and physics, but since we cannot presume any considerable familiarity with these, only problems involving the simplest geometrical principles have been inserted. While the elegance of algebraic methods is best seen in the solution and discussion of problems of equal generality with their algebraic statement, yet those we give are not entirely of this class.

PROBLEMS.

1. The hypothenuse of a right angled triangle is 10, and the excess of the perpendicular over the base is 2. Find the sides of the triangle.

2. The hypothenuse of a right angled triangle is h, and the excess of the perpendicular over the base is e. Find the sides of the triangle.

Can e in this problem be assigned *any value whatever?*

3. The perimeter of a rectangle is 16 feet, and its area is 15 square feet. Find the dimensions of the rectangle.

4. The perimeter of a rectangle is p feet, and its area is a square feet. Find the dimensions of the rectangle.

Show, from the result, that the square is the greatest possible rectangle which can be made with a given perimeter.

5. The sum of the squares of three consecutive odd numbers is 83. Find the numbers.

What would you say in case the number 56 was given in place of 83?

Make the problem read so that 56 will be allowable.

6. The sum of the squares of four consecutive even numbers is 120. Find the numbers.

7. If 962 men were drawn up in two squares, and it were found that one square had 18 more ranks than the other, what would be the size of each square?

8. A boat's crew row $3\frac{1}{2}$ miles down a river and back again in 1 hour and 40 minutes. Supposing the river to have a current of 2 miles per hour, find the rate at which the crew would row in still water.

What do you say about the negative result?

9. A boat's crew row d miles down a river and back again in t hours. Supposing the river to have a current of r miles per hour, find the rate of rowing in still water.

Show from the result that the problem will always give one positive and one negative value of x for all values of d, t or r.

10. The total area of two squares is a square feet. A side of one square is found to differ from a side of the other by d feet. Find the side of each square.

Is this problem possible for all values of d?

11. Two trains are dispatched from a station, one starting an hour before the other. The rate of motion of the later train is 5 miles per hour more than that of the other, and it overtakes the first train at a distance of 150 miles from the station. Find the rate of motion of each train.

12. Generalize the foregoing problem and solve it. Discuss the results.

13. A rectangular metal plate is 20 inches longer than wide. It is expanded by heat until each dimension increases by $\frac{1}{20}$ of its former length, thereby increasing the area of the plate 246 square inches. Find the original dimensions of the plate.

14. A man, born in 1806, died at the age of x in the year x^2. When did he die?

15. Two trains pass at a junction. One is traveling south at the rate of 30 miles an hour and the other is traveling west at the rate of 40 miles per hour. How long before the two trains are 100 miles apart?

Interpret the two results.

16. Two trains, A and B, are traveling on roads at right angles to each other, each approaching the crossing. A is 10 miles from the crossing and traveling uniformly 30 miles an hour, while at the same instant B is 20 miles from the crossing and traveling uniformly 40 miles an hour. When will they be 5 miles apart?

Explain the two results.

17. Two trains, A and B, are traveling on roads at right angles to each other. A is 40 miles from the crossing and is moving *towards* it at the uniform rate of 30 miles an hour. B is 20 miles from the crossing and is moving *from* it at the uniform rate of 25 miles an hour. At what times are the trains 90 miles apart?

Interpret the results.

18. Along the sides of a right angle two bodies, A and B, move with uniform velocity. A is a miles from the vertex and moving p miles per hour, while at the same instant B is b miles from the vertex and moving q miles per hour. At what times are the two bodies d miles apart?

Show that the result obtained can be used as a formula to solve Prob. 16.

Show that by giving the proper interpretation to q, as to its positive or negative character, that the formula can be made to solve either Prob. 16 or 17 at will.

Under what conditions will the bodies *never* be d miles apart?

19. Two circles, A and B, move with their centers always on the sides of a right angle. A, whose radius is R feet, is a feet from the vertex and moving uniformly p feet per second. B, whose radius is r feet, is b feet from the vertex and moving uniformly q feet per second. At what times are the circles tangent to each other?

Result: Tangent externally in

$$\frac{ap+bq\pm\sqrt{(R+r)^2(p^2+q^2)+(ap+bq)^2}}{p^2+q^2} \text{ seconds.}$$

Tangent internally in

$$\frac{ap+bq\pm\sqrt{(R-r)^2(p^2+q^2)+(ap+bq)^2}}{p^2+q^2} \text{ seconds.}$$

Show that it is possible for them to be tangent externally and not tangent internally.

Show that it is impossible for the circles to be tangent internally without first being tangent externally.

Show that the known quantities may have such values that the two circles will never be tangent at all.

20. Find the side of an equilateral triangle, knowing that a side exceeds the altitude by d feet.

CHAPTER V.

THEORY OF QUADRATIC EQUATIONS AND QUADRATIC FUNCTIONS.

1. It follows immediately from the definition (I Art. 4) that every rational integral quadradic function of x is of the form

$$lx^2+nx+r$$

where l, n and r stand for any algebraic numbers whatever, positive or negative, integral or fractional, commensurable or incommensurable.

If we take the typical quadratic equation

$$ax^2+bx=c$$

and transpose the c to the left-hand side of the equation it becomes

$$ax^2+bx-c=0.$$

This can obviously be said to be of the form

$$lx^2+nx+r=0$$

and consequently a quadratic equation may be defined as an equation which can be placed in the form of a rational integral quadratic function equal to zero.

Since a root of an equation has been defined as any expression which substituted for the unknown will satisfy the equation, therefore it is evident from the form

$$ax^2+bx-c=0$$

that a root of a quadratic equation may also be stated to be an expression which substituted for x causes ax^2+bx-c to equal zero; that is, causes the function* of x to vanish.

Hence we may say: *A quadratic equation is any equation which can be put in the form of a rational integral quadratic function equal to zero, and a root of it is any expressiou which, substituted for x, causes the function of x to vanish.*

Thus the equation $x^2-3x=10$, whose roots are 5 and -2, when placed in the form of a function of x equal to zero, becomes

$$x^2-3x-10=0.$$

It is now seen that the roots are such quantities that, when sub-

*Because of the array of adjectives in the expression ' rational integral quadratic function of x" we shall often, for the remainder of this chapter, use the expression "function of x" in its place.

stituted for x, cause the function of x to vanish. For the function of x is

$$x^2-3x-10$$

and putting 5 for x it becomes

$$25-15-10$$

which is zero. Putting -2 for x the function of x becomes

$$4+6-10$$

which is also zero.

If anything else than a root is put for x the function will not vanish; thus when

$x=-4$,	function of	x	becomes	$16+12-10=$	18
$x=-3$,	"	"	"	$9+9-10=$	8
[$x=-2$,	"	"	"	$4+6-10=$	0]
$x=-1$,	"	"	"	$1+3-10=$	-6
$x=0$,	"	"	"	$0+0-10=$	-10
$x=1$,	"	"	"	$1-3-10=$	-12
$x=2$,	"	"	"	$4-6-10=$	-12
$x=3$,	"	"	"	$9-9-10=$	-10
$x=4$,	"	"	"	$16-12-10=$	-6
[$x=5$,	"	"	"	$25-15-10=$	0]
$x=6$,	"	"	"	$36-18-10=$	8

2. If we suppose the quadratic function divided through by the coefficient of x^2 it may be represented by

$$x^2+ex+f.$$

If we take the quadratic $x^2+px=q$, and transpose the q to the other side of the equation, we obtain

$$x^2+px-q=0$$

where the left member is seen to be of the form x^2+ex+f. Then, since every quadratic may be put in the form $x^2+px=q$, it may also be placed in the form

$$x^2+px-q=0$$

or better

$$x^2+ex+f=0.$$

In either of the quadratic functions

$$lx^2+nx+r$$

or

$$x^2+ex+f$$

the term which does not contain x, that is r or f, is called the *absolute term*.

3. By solving the equation

$$x^2+ex+f=0$$

it will be found that its roots are

$$-\tfrac{1}{2}e+\sqrt{\tfrac{1}{4}e^2-f} \text{ and } -\tfrac{1}{2}e-\sqrt{\tfrac{1}{4}e^2-f}.$$

4. THEOREM. *Every quadratic function of x can be resolved into the product of two linear functions of x.*

Take the function of x in the form

$$x^2+ex+f.$$

add and subtract $\frac{1}{4}e^2$ from the function, thus not altering its value. We obtain then

$$x^2+ex+\tfrac{1}{4}e^2-\tfrac{1}{4}e^2+f.$$

This may be written

$$(x+\tfrac{1}{2}e)^2-(\tfrac{1}{4}e^2-f),$$

or, if we please, as the difference of two squares,

$$(x+\tfrac{1}{2}e)^2-(\sqrt{\tfrac{1}{4}e^2-f})^2.$$

Writing this as the product of the sum and difference, it takes the form

$$[(x+\tfrac{1}{2}e)-\sqrt{\tfrac{1}{4}e^2-f}]\,[(x+\tfrac{1}{2}e)+\sqrt{\tfrac{1}{4}e^2-f}]$$

or $$(x+\tfrac{1}{2}e-\sqrt{\tfrac{1}{4}e^2-f})(x+\tfrac{1}{2}e+\sqrt{\tfrac{1}{4}e^2-f}),$$

which is the product of two linear functions of x.

5. EXAMPLES. Resolve the following quadratic functions into the product of two linear functions of x:

1. $x^2-x-210$.

2. $3x^2+2x-85$.

3. $x^2-6bx+9b^2$.

4. $4a^2x^2-4ax+1$.

5. $x^2-14x+33$.

6. THEOREM. *If the roots of a quadratic equation are a and b, then the equation may always be put in the form $(x-a)(x-b)=0$.*

By Art. 4, the equation

$$x^2+ex+f=0 \qquad (1)$$

may always be placed in the form

$$(x+\tfrac{1}{2}e+\sqrt{\tfrac{1}{4}e^2-f})(x+\tfrac{1}{2}e-\sqrt{\tfrac{1}{4}e^2-f})=0. \qquad (2)$$

If we represent the two roots of (1) by a and b for the sake of brevity, we see from Art. 3, that

$$a=-\tfrac{1}{2}e+\sqrt{\tfrac{1}{4}e^2-f} \text{ and } b=-\tfrac{1}{2}e-\sqrt{\tfrac{1}{4}e^2-f}.$$

Substituting these in equation (2), it becomes

$$(x-a)(x-b)=0.$$

7. COROLLARY. *If all the terms of a quadratic be transposed to one side, that member is exactly divisible by x minus a root.*

8. COROLLARY. *The form $(x-a)(x-b)=0$ may be used interchangeably with $x^2+ex+f=0$ to represent any quadratic equation.*

9. THEOREM. *Every quadratic equation with one unknown quantity has two roots and only two.*

It has been shown that every quadratic equation can be placed in the form

$$(x-a)(x-b)=0.$$

This equation is satisfied when the left member is zero. But the left member becomes zero when either one of its two factors is zero; that is, when $x=a$ or $x=b$. Because each of these two values of x satisfies the equation it has *two* roots. But the equation can have no other root; for if any other value than a or b be assigned to x, neither of the factors will be zero, and consequently their product will not be zero. Hence there can be *no more* than two roots.

It is not claimed that there must be two *different* roots. In fact, there is nothing in any of the reasoning thus far which shows that a and b must always have different values. In general, they are different from each other, but a special case would be where they are alike. In this case the quadratic takes the form

$$(x-a)(x-a)=0,$$

and we still speak of *two* roots because there are *two* factors and because it is merely a special case of the general truth. To say that an equation has two roots equal to each other is merely another way of saying that there is but one value which satisfies the equation.

10. THEOREM. *When a quadratic equation is in the form $x^2+ex+f=0$, the coefficient of x with its sign changed equals the sum of the two roots, and the absolute term equals the product of the two roots.*

The two roots of the equation

$$x^2+ex+f=0$$

are $-\frac{1}{2}e+\sqrt{\frac{1}{4}e^2-f}$

and $\underline{-\frac{1}{2}e-\sqrt{\frac{1}{4}e^2-f}}$

$$-\ e+\ 0$$

Adding them, their sum is seen to be $-e$, or the coefficient of x with its sign changed.

Multiplying the two roots together, recognizing the product of a sum and difference, we obtain

$$\left(-\tfrac{1}{2}e+\sqrt{\tfrac{1}{4}e^2-f}\right)\left(-\tfrac{1}{2}e-\sqrt{\tfrac{1}{4}e^2-f}\right)=\left(-\tfrac{1}{2}e\right)^2-\left(\sqrt{\tfrac{1}{4}e^2-f}\right)^2$$
$$=\tfrac{1}{4}e^2-\tfrac{1}{4}e^2+f=f,$$

which is the absolute term of the equation.

ANOTHER METHOD.

(a). First, suppose the two roots not equal to each other. Call, for abbreviation, the two roots of the equation $x^2+ex+f=0$ a and b. Then, by the definition of a root, we have

$$a^2+ea+f=0 \qquad (1)$$

and

$$b^2+eb+f=0. \qquad (2)$$

Subtracting *(2)* from *(1)* we obtain

$$a^2-b^2+e(a-b)=0, \qquad (3)$$

or, dividing through by $a-b$,

$$a+b+e=0,$$

or

$$e=-(a+b). \qquad (4)$$

That is, the coefficient of x is the sum of the roots with opposite signs.

Now substitute this value of e in equation *(1)*. It becomes

$$a^2-a(a+b)+f=0, \qquad (5)$$

or

$$-ab+f=0, \qquad (6)$$

whence

$$f=ab. \qquad (7)$$

That is, the absolute term is equal to the product of the two roots.

(b). If the two roots equal each other, that is, if each is equal to a, the form $(x-a)(x-b)=0$ becomes $(x-a)(x-a)=0$, or

$x^2-2ax+a^2=0$, where it is seen that $-2a$ is the sum of two roots with the opposite sign, and a^2 equals their product.

11. EXAMPLES. We can now form a quadratic equation which shall have any two roots we desire. Suppose we wish a quadratic whose roots shall be 3 and 5. Then $e=-(3+5)=-8$, and $f=3\times 5=15$. Then the equation is

$$x^2-8x+15=0.$$

1.	Form the equation whose roots shall be	3 and -5.
2.	" " " " " "	-3 and 5.
3.	" " " " " "	-3 and -5.
4.	" " " " " "	a and $\frac{1}{a}$.
5.	" " " " " "	$2+\sqrt{3}$ and $2-\sqrt{3}$
6.	" " " " " "	$\sqrt{7}$ and $-\sqrt{7}$.
7.	" " " " " "	-5 and 0.
8.	" " " " " "	6 and 6.
9.	" " " " " "	0 and 0.

12. The student should not understand that there is only *one* method of solving the quadratic equation. The fact is that the result may be reached in a great variety of ways, that of IV, Art. 9, merely being one among a great number. But many of the different methods that have been proposed are, in the last analysis, essentially the same, and they all resolve themselves into the one principle of reducing the quadratic to some form of a simple equation. We give a few methods of solution to show the student what a variety of means may be made use of in such work.

(*a*). *By reduction to an incomplete quadratic.*

$$x^2+ex+f=0.$$

Suppose $x=y-\frac{1}{2}e$, where y is a new unknown quantity; then the equation becomes

$$(y-\tfrac{1}{2}e)^2+e(y-\tfrac{1}{2}e)+f=0,$$

or

$$y^2-ey+\tfrac{1}{4}e^2+ey-\tfrac{1}{2}e^2+f=0,$$

or

$$y^2-\tfrac{1}{4}e^2+f=0.$$

This is an equation of the first degree in terms of y^2. Solving we obtain

$$y^2=\tfrac{1}{4}e^2-f,$$

whence $$y=\pm\sqrt{\tfrac{1}{4}e^2-f},$$

and since $x=y-\frac{1}{2}e$,

$$x=-\tfrac{1}{2}e\pm\sqrt{\tfrac{1}{4}e^2-f}.$$

Solve in this manner the equation $x^2-5x-14=0$.

(b). By considering the quadratic as the product of two linear factors.

Suppose the function of x to be the product of two factors of the form $(x+\frac{1}{2}e+u)(x+\frac{1}{2}e-u)$, where u is a new unknown quantity. Then we have the equation

$$x^2+ex+f=(x+\tfrac{1}{2}e+u)(x+\tfrac{1}{2}e-u).$$

Expanding the right member of the equation we obtain

$$x^2+ex+f=x^2+ex+\tfrac{1}{4}e^2-u^2.$$

Therefore $$u^2=\tfrac{1}{4}e^2-f,$$

whence $$u=\pm\sqrt{\tfrac{1}{4}e^2-f}$$

Now as the product of the two factors $(x+\frac{1}{2}e+u)(x+\frac{1}{2}e-u)$ must equal zero we must take x either $-\frac{1}{2}e-u$ or $-\frac{1}{2}e+u$. Take the former, and

$$x=-\tfrac{1}{2}e-u=-\tfrac{1}{2}e\mp\sqrt{\tfrac{1}{4}e^2-f}.$$

Solve by this method the equation $x^2-6x=16$.

(c). By the sum and product of the roots.

Suppose the two roots of $x^2+ex+f=0$ to be y and z. Then we know by Art. 10,

$$y+z=-e \qquad (1)$$

and $$yz=f, \qquad (2)$$

squaring (1) we obtain

$$y^2+2yz+z^2=e^2. \qquad (3)$$

Subtracting four times (2) from this

$$y^2-2yz+z^2=e^2-4f,$$

or, extracting the root, $y-z=\pm\sqrt{e^2-4f}$,

and since $$y+z=-e,$$

$$y=\frac{-e\pm\sqrt{e^2-4f}}{2}$$

$$z=\frac{-e\mp\sqrt{e^2-4f}}{2}$$

Solve in this manner the equation $3x^2-5x+2=0$.

13. DISCRIMINATION OF THE ROOTS OF THE QUADRATIC EQUATION. The roots of the equation

$$x^2+ex+f=0$$

are $\qquad x=-\frac{1}{2}e+\sqrt{\frac{1}{4}e^2-f}$ and $x=-\frac{1}{2}e-\sqrt{\frac{1}{4}e^2-f}$.

(a). If $\frac{1}{4}e^2-f$ is positive there are two real and unequal roots.

(b). If $\frac{1}{4}e^2-f$ is negative there are two imaginary roots.

(c). If $\frac{1}{4}e^2-f$ is zero the two values of x each reduce to $-\frac{1}{2}e$ and the two values of x are real and equal.

(d). If $\frac{1}{4}e^2-f$ is a perfect square the two roots are rational, if e is rational.

(e). If $\frac{1}{4}e^2-f$ is not a perfect square the roots are irrational.

The expression $\frac{1}{4}e^2-f$ is called the Discriminant.

The case where $\frac{1}{4}e^2-f$ is zero deserves further attention. If $\frac{1}{4}e^2-f=0$ then $\frac{1}{4}e^2=f$ and the equation $x^2+ex+f=0$ becomes

$$x^2+ex+\tfrac{1}{4}e^2=0$$

or

$$(x+\tfrac{1}{2}e)(x+\tfrac{1}{2}e)=0.$$

Whence we see that when a quadratic equation has two equal roots the function of x is a complete square.

14. TO FIND THE CONDITIONS THAT A QUADRATIC EQUATION MAY HAVE TWO POSITIVE ROOTS. Represent the roots by a and b.

Then since $\qquad -(a+b)=e$

if the roots are both positive the coefficient of x must be negative.

Also since $\qquad ab=f$

if the roots are both positive the absolute term must be positive.

Hence *the full condition that both the roots of a quadratic be positive is that the coefficient of x be negative and the absolute term positive.*

15. TO FIND THE CONDITION THAT A QUADRATIC EQUATION MAY HAVE TWO NEGATIVE ROOTS. Represent the roots as before.

Then since $\qquad -(a+b)=e$

if both roots are negative the coefficient of x must be positive.

And since $\qquad ab=f$

if both roots are negative, the absolute term must be positive.

Hence *the full condition that both the roots of a quadratic be negative is that the coefficient of x be positive and the absolute term negative.*

16. TO FIND THE CONDITION THAT A QUADRATIC EQUATION MAY HAVE ONE POSITIVE AND ONE NEGATIVE ROOT.

Since $$ab=f$$
if the roots are of opposite signs the absolute term must be negative.

Since $$-(a+b)=e$$
if the positive root is numerically the greater, e is negative and in case the negative root is numerically the greater, e will be pcsitive.

The condition that a quadratic have roots of opposite signs is merely that the absolue term be negative, but if the coefficient of x is negative the positive root is numerically the greater and if the coefficient of x is positive the negative root is numerically the greater.

17. EXAMPLES. Discriminate the roots of the following equations; that is, tell by inspection whether the roots are real or imaginary, and if real, tell whether they are positive or negative.

1. $x^2+8x-9=0$.

2. $x^2+70x+1200=0$.

3. $x^2-4x+4=0$.

4. $x^2+10x+45=0$.

5. $x^2-8x+20=0$.

6. $x^2=10x-25$.

7. $x^2-12x=-27$.

8. $\frac{1}{2}x^2-\frac{1}{3}x=\frac{5}{8}$.

18. In a manner similar to that of Arts. 14—16 the student may determine the following:

1. Find the condition that a quadratic equation may have two roots numerically equal but of opposite signs.

2. Find the condition that a quadratic equation may have two roots which are reciprocals of each other.

3. Find the condition that a quadratic equation may have one root equal to zero.

19. MISCELLANEOUS EXERCISES IN THE THEORY OF QUADRATICS.

1. If a and b are the roots of $x^2+ex+f=0$, find the value of a^2+b^2 in terms of e and f.

$$a+b=-e$$
$$ab=f$$

whence $\qquad a^2+2ab+b^2=e^2$

and $\qquad 2ab=2f.$

Therefore $\qquad a^2+b^2=e^2-2f.$

2. Find the value of $\frac{1}{a}+\frac{1}{b}$ in terms of e and f.

$$\frac{1}{a}+\frac{1}{b}=\frac{a+b}{ab}=-\frac{e}{f}.$$

3. Prove $(a-b)^2=e^2-4f$.

4. Find the value of $\frac{a}{b}+\frac{b}{a}$ in terms of e and f.

5. Given the equation $x^2+ex+f=0$, form the equation whose roots are the squares of the roots of this equation.

If the roots of this equation be called a and b, the roots of the required equation will be a^2 and b^2. The coefficient of x must then be $\qquad a^2+b^2$

or, by Ex. 1, $\qquad e^2-2f.$

The absolute term must be $\qquad a^2b^2$

or $\qquad f^2.$

Hence the required equation must be

$$x^2+(e^2-2f)x+f^2=0.$$

6. Form an equation whose roots shall be the reciprocals of the roots of $x^2+ex+f=0$.

7. Prove that the equation $x^2-kx-d^2=0$ cannot have imaginary roots.

8. Find the value of m such that the roots of $x^2+ex+m=0$ will differ by 2.

CHAPTER VI.

SINGLE EQUATIONS.

1. Every equation containing one unknown quantity can be put in the form

$$\textit{Function of } x=0$$

by transposing all the terms to the left side of the equation.

If it is an equation of the first degree it will always reduce to the form

$$x-a=0,$$

where a must stand for any quantity whatever, positive or negative, integral or fractional, commensurable or incommensurable. It is evident that this equation has the root a and no other. An equation of the first degree might be defined as an equation which can be placed in the form of a rational integral linear function of x equal to zero.

We have seen that every quadratic equation can be placed in the form

$$(x-a)(x-b)=0,$$

which has the two roots a and b and no others. Thus every quadratic equation can be placed in the form of the product of two rational integral linear functions of x equal to zero.

It will be proved in Part II that every cubic equation can be put in the form

$$(x-a)(x-b)(x-c)=0,$$

and that it has three roots, a, b, and c, and no others. That is, every cubic equation can be placed in the form of the product of three rational integral linear functions of x equal to zero.

It will also be shown that an equation of the fourth degree can be thrown in the form

$$(x-a)(x-b)(x-c)(x-d)=0.$$

These and other important properties of equations containing one unknown quantity were first discovered by Vieta (1540—1603), but were independently and more elaborately treated by Harriot (1560—1621).

2. We are led to inquire what operations can be performed upon the members of an equation without modifying the values of the unknown. Now, by the principles of algebra, *an equation remains true* if we unite the same quantity to both sides by addition or subtraction; or if we multiply or divide both members by the same quantity; or if like powers or roots of both members be taken. But, as hinted in IV, Art. 2, *these operations may affect the value of the unknown*. Thus the roots of the equation

$$3(x-5)=x(x-5)+x^2-25 \qquad (1)$$

are -1 and 5. Either of these when substituted for x will satisfy the equation. But divide the equation through by $x-5$. The resulting equation is

$$3=x+x+5. \qquad (2)$$

Now this equation is *not* satisfied for $x=5$. The sole root is -1. Hence, although equation (2) must be *true* if (1) is, yet the equations are not equivalent, since their solutions are not identical. One root has disappeared in the transformation. Just how this occurs will be best seen after we place (1) in the form $(x-a)(x-b)=0$. Since the roots of (1) are -1 and 5, by the principle of V, Art. 6 it is equivalent to

$$(x-5)(x+1)=0. \qquad (3)$$

Now, if we divide this through by $x-5$, we remove that factor in the left member which is zero for $x=5$. Consequently the equation will be no longer satisfied for $x=5$. If we should divide through by $x+1$ the equation will be no longer satisfied for $x=-1$.

Also consider the equation

$$x^2-6x+8=0. \qquad (4)$$

It is satisfied for $x=2$ or $x=4$. Now multiplying both members by $x+3$ we obtain

$$(x+3)(x^2-6x+8)=0. \qquad (5)$$

But this equation is satisfied for either $x=-3$, or $x=2$, or $x=4$. Hence, although multiplying both members of (4) by $x+3$ has not altered the equality, yet a value of x extraneous to the original equation has been introduced.

Again the equation

$$2x-1=x+5 \qquad (6)$$

is satisfied only by the value $x=6$. Now square both sides of the equation, obtaining

$$4x^2-4x+1=x^2+10x+25, \qquad (7)$$

which is satisfied for either $x=6$ or $x=-\frac{8}{3}$. Here, obviously, an extraneous solution has been introduced by the operation of squaring both members.

In a like manner notice the effect of taking a root of both members of an equation. Thus suppose

$$4x^2=(x-6)^2. \qquad (8)$$

This is satisfied for either $x=2$ or -6. Take the square root of each member and we obtain

$$2x=x-6, \qquad (9)$$

which is satisfied only by $x=-6$. We have lost one of the solutions of the equation during this transformation, Equation (8) is really not equivalent to (9), but to the two equations

$$\left\{\begin{array}{l} 2x=+(x-6) \\ 2x=-(x-6) \end{array}\right\} \qquad (10)$$

We have given examples enough to show that certain operations upon an equation may modify the solution. Thus we see that during a series of transformations which sometimes an equation must undergo before we can reach the values of the unknown it is possible that the solutions that satisfy the original equation may all be lost and that any number of new ones may be introduced, so that the final results may have no relation at all to the problem in hand. It is now proposed to formulate certain propositions which will enable us to tell the exact place in the process of any solution where roots may be lost or new ones may enter. We will then be able to perform the different operations on the members of an equation if we will note at the time their effect on the solution and finally make allowance for it in the result. This fact must be emphasized: *the test for any solution of an equation is that it satisfy the original equation.* "No matter how elaborate or ingenious the process by which the solution has been obtained, if it do not stand this test it is no solution; and, on the other hand, no matter how simply obtained, provided it do stand this test, it is a solution."—*Chrystal.*

When one equation is derived from another by an operation which has no effect one way or another on the solution, it may be

spoken of as a legitimate transformation or derivation; when the operation does have an effect upon the final result, it may be called a *questionable* derivation, meaning thereby that the operation requires examination.

If there are two equations such that any solution of the first is a solution of the second, and also that any solution of the second is a solution of the first, the two equations are said to be *equivalent*.

3. THEOREM. *The transformation of an equation by the addition or subtraction from both members of either a known quantity or a function of the unknown is a legitimate derivation.*

An equation containing one unknown quantity, as it commonly appears with quantities on each side of the equation, may be generalized in thought by the expression

$$A \text{ function of } x = \text{Another function of } x.$$

Or, using L to represent the left-hand side of the equation, whatever it may be, and R to represent the expression on the right-hand side, we can represent any equation very conveniently by

$$L=R. \qquad (1)$$

Now suppose that T, which may be either a known quantity or a function or the unknown, be added to both members of the equation, making

$$L+T=R+T. \qquad (2)$$

Now it is plain that (2) cannot be satisfied unless $L=R$ and that it *is* satisfied if $L=R$. Hence (2) means no more nor less than (1). Therefore the derivation is legitimate.

4. COROLLARY. *Transposition of terms from one member to the other, changing the signs at the same time, is legitimate.* Thus if $L=R$, to pass to $L-R=0$ is merely subtracting R from both members.

5. THEOREM. *Multiplying both members of an equation by the same expression is legitimate if the expression is a known quantity, but questionable if the expression is a function of the unknown.*

Represent the equation by

$$L=R. \qquad (1)$$

Multiply both members by T, obtaining

$$LT=RT. \qquad (2)$$

Now this may be written as

$$(L-R)T=0. \qquad (3)$$

If T is a known quantity this can only be satisfied by the supposition that $L=R$, that is, the equation is equivalent to (1). But if T is a function of the unknown (for example, $2x$ or $x+5$, or x^3+8) then (3) may be satisfied by any value of the unknown that will make $T=0$ (such as $x=0$, or $x=5$, or $x=-2$, respectively, in the three examples given), whence (3) would not be equivalent to (1) but to the two equations.

$$\left\{\begin{matrix} L=R \\ T=0. \end{matrix}\right\}$$

6. Corollary. *If any equation involves fractions with only known quantities in the denominators, it is legitimate to clear of fractions.* The multiplier in this case is a known quantity.

7. Theorem. *If an equation involves irreducible fractions with unknown quantities in the denominators, and the denominators are all prime to each other, it is legitimate to integralize by multiplying through by the least common multiple of the denominators.*

To illustrate the reasoning take the equation

$$\frac{A}{X_1}+\frac{B}{X_2}+\frac{C}{X_3}=0. \qquad (1)$$

where the fractions are supposed to be in their lowest terms and X_1, X_2, X_3 represent *different* functions of the unknown and where A, B and C are either known quantities or functions of the unknown. Multiplying by the least common multiple of the denominators we obtain

$$AX_2X_3+BX_1X_3+CX_2X_3=0. \qquad (2)$$

Now, since X_1, X_2 and X_3 are prime to each other no common factor has been introduced by multiplying by $X_1X_2X_3$, and consequently no additional solutions can appear.

8. As an example under the above theorem take the equation

$$\frac{7-x}{11-2x}+\frac{4x-5}{3x-1}=2. \qquad (1)$$

These fractions are in their lowest terms and their denominators are prime to each other. The least common multiple of the denominators is $(11-2x)(3x-1)$. Multiplying through by this we obtain

$$(3x-1)(7-x)+(11-2x)(4x-5)=2(11-2x)(3x-1). \qquad (2)$$

Now we can see that although (1) has been multiplied through both by $(11-2x)$ and $(3x-1)$, yet neither of these has been introduced as a factor tbrough the equation. Hence there is no additional solution introduced. The roots of (2) will in fact be found to be 4 or -10, which values also satisfy (1).

But an extraneous solution may be introduced if the denominators are not prime to each other, or if some of the fractions are not in their lowest terms. Thus

$$\frac{3x}{x-3}=\frac{6}{x+3}+\frac{9}{x-3} \qquad (3)$$

has two denominators alike, and consequently not prime to each other. Multiplying through by the common denominator x^2-9, we obtain

$$3x(x+3)=6(x-3)+9(x+3) \qquad (4)$$

or, reducing,

$$x^2-2x=4 \qquad (5)$$

whose roots are 3 and -1. Now if we put the original equation (3) in the form

$$\frac{3x-9}{x-3}=\frac{6}{x+3}.$$

that is

$$3=\frac{6}{x+3} \qquad (6)$$

it is seen that it is satisfied only for $x=-1$. Hence a solution was introduced in clearing (3) of fractions. It is easy to see that (3) is really equivalent to (6) and hence that in clearing (3) of fractions by multiplying by x^2-9 we multiplied by $x-3$ when it was not necessary; this is where the solution $x=3$ was introduced.

9. THEOREM. *Every equation can be integralized legitimately.* For if the several fractions in the equation are not in their lowest terms they can be so reduced. Then these fractions can all

be transposed to one side of the equation, their common denominator found and then added together. This will now give but one fraction in the equation, and, when this is reduced to its lowest terms, we will have an equation of the form

$$\frac{N}{D}=R$$

which, since $\frac{N}{D}$ *is in its lowest terms* by supposition, will take on no additional solutions when multiplied through by D, according to Art. 7.

10. THEOREM. *The raising of both members of an equation to the same power is equivalent to multiplying through by a function of the unknown and hence is a questionable derivation.*

Take the equation

$$L=R \qquad (1)$$

and raise both members to the nth power, obtaining

$$L^n=R^n. \qquad (2$$

Now (1) is equivalent to

$$L-R=o$$

and (2) is equivalent to

$$L^n-R^n=o. \qquad (4)$$

But (4) can be derived from (3) by multiplying both members by

$$L^{n-1}+L^{n-2}R+L^{n-3}R^2+\ldots+L^2R^{n-3}+LR^{n-2}+R^{n-1}$$

whence (2) is equivalent to the *two* equations

$$\begin{cases} L=R \\ L^{n-1}+L^{n-2}R+L^{n-3}R^2+\ldots+L^2R^{n-3}+LR^{n-2}+R^{n-1}=0. \end{cases}$$

11. THEOREM. *Dividing both members of an equation by the same expression is legitimate if the expression is a known quantity, but questionable if it is a function of the unknown.*

Suppose both members of the equation to be divisible by T and write the equation

$$LT=RT. \qquad (1)$$

Now if T is a known quantity, then by Art. 5 this equation is equivalent to

$$L=R \qquad (2)$$

whence division by T would be legitimate. But if T is a function of the unknown quantity, then (1) is equivalent to the two equations

$$\begin{cases} L=R \\ T=0. \end{cases}$$

Division by T would give us but one of these and consequently solutions would be lost. Hence the division by a function of the unknown is a questionable derivation.

12. THEOREM. *The extraction of the same root of both members of an equation is equivalent to dividing by a function of the unknown and hence is a questionable derivation.*

For we can pass from

$$L^n=R^n \qquad (1)$$

to

$$L=R \qquad (2)$$

by dividing both members of (1) by

$$L^{n-1}+L^{n-2}R+L^{n-3}R^2+\ .\ .\ .\ +L^2R^{n-3}+LR^{n-2}+R^{n-1}.$$

Hence, by Art. 11, root extraction is a questionable derivation.

13. EXAMPLES OF THE INTEGRALIZING OF EQUATIONS. In the following equations the student should note the precise effect of all questionable operations at the time they are performed.

1. Solve $\frac{1}{2}(x-5)(x-3)(x-2)=(x-5)(x+3)(x+2)$.

2. Solve $\dfrac{x+2}{x-2}-\dfrac{x-2}{x+2}=\dfrac{5}{6}$.

3. Solve $\dfrac{1}{2(x-1)}+\dfrac{3}{x^2-1}=\dfrac{1}{4}$.

4. Solve $\dfrac{1}{x}+\dfrac{2}{x}=\dfrac{x+1}{x}$.

5. Solve $\dfrac{x^2}{4(x^2-1)}=1-\dfrac{x}{x^2-1}-\dfrac{1}{x^2-1}$.

6. Solve $\dfrac{x}{x-1}+\dfrac{x+1}{x}=\dfrac{13}{6}$.

7. Solve $\dfrac{x^2+2x-15}{x^2-x-6}=\dfrac{x+5}{x+2}$.

14. EXAMPLES OF THE RATIONALIZATION OF EQUATIONS. The most expeditious method for rationalizing any given equation depends upon the peculiar make up of the equation, and can only be determined by the student after a little experience with this class of equations.

1. Given $\sqrt{9+x}+x=11$. *(1)*

Transpose everything but the radical to the right-hand side of the equation and we obtain

$$\sqrt{9+x}=11-x. \qquad (2)$$

Squaring both sides gives

$$9+x=121-22x+x^2 \qquad (3)$$

and solving this quadratic we find

$$x=7 \text{ or } 16.$$

From *(2)* to *(3)* is a questionable derivation; for squaring both members of an equation, $L=R$, we have found *(*Art. 10*)* to be equivalent to multiplying through by $L+R$, and that the resulting equation is equivalent to the two equations

$$\begin{cases} L=R \\ L+R=0. \end{cases}$$

Therefore *(3)* is equivalent to the two equation

$$\begin{Bmatrix} \sqrt{9+x}=11-x \\ \sqrt{9+x}+11-x=0 \end{Bmatrix} \qquad (4)$$

or to

$$\begin{Bmatrix} +\sqrt{9+x}+x=11 \\ -\sqrt{9+x}+x=11 \end{Bmatrix} \qquad (5)$$

Hence, if we understand equation *(1)* to read

The positive square root of $(9+x)+x=11$

then a new solution has been introduced between *(2)* and *(3)*. But if we understand equation *(1)* to read

A square root of $(9+x)+x=11$

then it is equivalent to both the equations in *(5)*, and no solution has been introduced. This is because the introduced equation, $\pm L+R=0$ is identical with the original equation $\pm L=R$.

In these cases the student will always find that *rationalization may or may not be considered as a questionable derivation according as we consider the radicals to call for a* PARTICULAR *root or* ANY *root of the expressions involved.*

It is more in accordance with the generalizing spirit of algebra to consider the radical sign, wherever it occurs, as calling for any of the possible roots. This will be better appreciated by the student when he learns in Part II that every expression has three different cube roots, four fourth roots, five fifth roots, etc.

2. Solve $\sqrt{x}+\sqrt{x+6}=3.$ (1)

Squaring each side of the equation, obtain

$$x+2\sqrt{x^2+6x}+x+6=9. \qquad (2)$$

Transposing all but the radical to the right-hand side this becomes

$$2\sqrt{x^2+6x}=3-2x. \qquad (3)$$

Squaring, we obtain

$$4x^2+24x=9-12x+4x^2 \qquad (4)$$

or

$$x=\tfrac{1}{4}.$$

What are the questionable steps? What is their effect?

The above solution is really equivalent to the following:

$$\sqrt{x}+\sqrt{x+6}=3.$$

Transpose the 3 to left member, obtaining

$$\sqrt{x}+\sqrt{x+6}-3=0;$$

Multiplying through by the rationalizing factor of left member, (III, Art. 26) we obtain

$$(\sqrt{x}+\sqrt{x+6}-3)(\sqrt{x}+\sqrt{x+6}+3)$$
$$\times(\sqrt{x}-\sqrt{x+6}-3)(\sqrt{x}-\sqrt{x+6}+3)=0,$$

which reduces to

$$4x-1=0, \qquad (5)$$

or

$$x=\tfrac{1}{4}.$$

The introduced equations are

$$\sqrt{x}+\sqrt{x+6}+3=0$$
$$\sqrt{x}-\sqrt{x+6}-3=0.$$
$$\sqrt{x}-\sqrt{x+6}+3=0.$$

Here, then, is an apparent paradox: *three* solutions seem to have been introduced, yet there is only *one* in all!

This can be explained in the following manner. If we regard the radical signs as calling for *any one* of the two roots of the expression underneath, then the introduced equations are all identical with the original equation and hence could not give rise to a *different* solution. If we restrict ourselves to using that square root in each case which has the sign given before the radical, then none of the introduced equations *have any solution whatever*, and hence no solution is introduced in this case.

3. Solve $\dfrac{\sqrt{x}+\sqrt{a+x}}{\sqrt{x}-\sqrt{a+x}}=x.$

Sometimes such equations are best simplified by first rationalizing either the numerator or denominator of the fraction. Rationalizing the denominator of this fraction, the equation becomes

$$\frac{2x+a+2\sqrt{ax+x^2}}{x-a-x}=x,$$

or $$2\sqrt{ax+x^2}=-(ax+2x+a),$$

whence $$4ax+4x^2=(ax+2x+a)^2,$$

etc.

4. Solve $x+\sqrt{x+3}=4x-1$.

5. Solve $\sqrt{14-x}+\sqrt{11-x}=\dfrac{3}{\sqrt{11-x}}$.

6. Solve $\sqrt{4x+9}-\sqrt{x-1}=\sqrt{x+6}$.

7. Solve $\sqrt{x-9}+\sqrt{x+12}=\sqrt{x-4}+\sqrt{x+13}$.

8. Solve $\sqrt{x+a^2}-\sqrt{x}=b$.

9. Solve $\dfrac{20x}{\sqrt{10x-9}}-\sqrt{10x-9}=\dfrac{18}{\sqrt{10x-9}}+9$.

10. Solve $\sqrt{6x+4+\sqrt{x^4+10x^2+3x+10}}=x+3$.

11. Solve $\dfrac{1+2\sqrt{4x-7}}{6+5\sqrt{4x-7}}=\dfrac{3-4\sqrt{4x-7}}{3-10\sqrt{4x-7}}$.

12. Solve $\dfrac{\sqrt[3]{1+x^2}+\sqrt[3]{1-x^2}}{\sqrt[3]{1+x^2}-\sqrt[3]{1-x^2}}=1$.

13. Rationalize $\sqrt[3]{x^2}-\sqrt{x^3}=0$.

14. Solve $\dfrac{\sqrt{x}+\sqrt{x-3}}{\sqrt{x}-\sqrt{x-3}}=\dfrac{3}{x-3}$.

Criticise the following solution:

Multiply both terms of the first fraction by $\sqrt{x}+\sqrt{x-3}$, and we have

$$\frac{(\sqrt{x}+\sqrt{x-3})^2}{x-(x-3)}=\frac{3}{x-3}, \qquad (2)$$

or $$(\sqrt{x}+\sqrt{x-3})^2=\frac{9}{x-3}. \qquad (3)$$

Extracting the square root,

$$\sqrt{x}+\sqrt{x-3}=\frac{3}{\sqrt{x-3}}. \qquad (4)$$

Clearing of fractions,

$$\sqrt{x^2-3x}+x-3=3; \qquad (5)$$

whence

$$x=4. \qquad (6)$$

15. Solve $x=\dfrac{2}{x+\sqrt{2+x^2}}+\dfrac{2}{x-\sqrt{2-x^2}}$.

16. Solve $\sqrt{x}+\sqrt{2+x}=\dfrac{4}{\sqrt{2+x}}$.

17. Solve $\sqrt{5x+10}=\sqrt{5x}+2$.

18. Solve $\sqrt{1+x+\sqrt{1+x+\sqrt{1+x}}}=\sqrt{1-x}$.

19. Solve $\dfrac{x+\sqrt{x^2-9}}{x-\sqrt{x^2-9}}=(x+2)^2$.

Rationalize the denominator of the fraction.

15. EQUATIONS WHICH CAN BE SOLVED AS QUADRATICS. The method employed in the solution of quadratic equations will sometimes enable us to solve equations of other degrees, or even irrattonal equations. Thus consider the equation

$$3x^4-5x^2+4=2. \qquad (1)$$

Multiply both members by four times coefficient of x^4 and add the square of the coefficient of x^2 to each side, as in the solution of a quadratic equation. It then becomes

$$36x^4-60x^2+25=1. \qquad (2)$$

The left-hand member is now a perfect square. Whence, extracting the square root of both members, equation (2) becomes

$$6x^2-5=\pm 1$$

whence

$$x^2=1 \text{ or } \tfrac{2}{3}.$$

Therefore

$$x=+1 \text{ or } -1 \text{ or } +\sqrt{\tfrac{2}{3}} \text{ or } -\sqrt{\tfrac{2}{3}}.$$

As another example consider the equation

$$4\sqrt{x}+3x=4. \qquad (1)$$

Put $y=\sqrt{x}$, whence it is seen that (1) becomes

$$4y+3y^2=4. \qquad (2)$$

Solving this quadratic we find

$$y=\tfrac{2}{3} \text{ or } -2.$$

Whence, since $y=\sqrt{x}$

$$\sqrt{x}=\tfrac{2}{3} \text{ or } -2.$$

Therefore $x=\frac{4}{9}$ or 4.

These examples suggest the following theorems:

16. THEOREM. *Any equation which can be placed in the form $x^{2n}+ex^n+f=0$ can be solved as a quadratic.*

$$x^{2n}+ex^n+f=0 \qquad (1)$$

may be written

$$(x^n)^2+e(x^n)+f=0, \qquad (2)$$

which, if we regard (x^n) as the unknown qantity, is seen to be in the quadratic form. Completing the square of (2) it becomes

$$(x^n)^2+e(x^n)+\tfrac{1}{4}e^2=\tfrac{1}{4}e^2-f \qquad (3)$$

Whence $$x^n+\tfrac{1}{2}e=\pm\sqrt{\tfrac{1}{4}e^2-f}$$

or $$x^n=-\tfrac{1}{2}e\pm\sqrt{\tfrac{1}{4}e^2-f}.$$

Therefore $$x=\left(-\tfrac{1}{2}e\pm\sqrt{\tfrac{1}{2}e^2-f}\right)^{\frac{1}{n}} \qquad (4)$$

which is the solution of the equation of the proposed form.

17. THEOREM. *Any equation which can be placed in the form $X^{2n}+eX^n+f=0$, where X stands for any linear or quadratic function of the unknown, can be solved as a quadratic.*

For, by the last article, it will be found that

$$X=\left(-\tfrac{1}{2}e\pm\sqrt{\tfrac{1}{4}e^2-f}\right)^{\frac{1}{n}} \qquad (1)$$

Now, if X is a linear function of x, this equation is of the form

$$ax+b=\left(-\tfrac{1}{2}e\pm\sqrt{\tfrac{1}{4}e^2-f}\right)^{\frac{1}{n}} \qquad (2)$$

which can be easily solved for x.

If X is a quadratic function of x equation (1) must be of the form

$$ax^2+bx+c=\left(-\tfrac{1}{2}e\pm\sqrt{\tfrac{1}{4}e^2-f}\right)^{\frac{1}{n}} \qquad (3)$$

Now this is a quadratic equation in terms of x, since all other quantities in the equation are known, and hence the equation can be solved.

In treating examples which come under these two theorems it may be possible that we will not find *all* the values that will satisfy the given equation. This happens because we are not always able to find n different n th roots of a quantity, while that number really do exist. Thus from the equation

$$x^6+19x^3=216 \qquad (1)$$

we will find by considering x^3 the unknown quantity that

$$x^3=27 \text{ or } -8$$

whence $$x=3 \text{ or } -2.$$

But really 27 and -8 have each *three* different cube roots instead of merely the ones we have written above. The full consideration of this matter involves subjects somewhat more advanced, and more than the mere statement above given will not be attempted until Part II of the present work is reached.

18. EXAMPLES. The following five are examples under the theorem of Art. 16:

1. Solve $x^4+16x^2=225$.

2. Solve $x^6-9x^3+8=0$.

3. Solve $6x^4-35=11x^2$.

4. Solve $x^{\frac{2}{3}}+\frac{5}{2}=\frac{13}{4}x^{\frac{1}{3}}$.

5. $\sqrt[4]{x^3}-2\sqrt{x}+x=0$.

The following five are examples under the theorem of Art. 17.

6. Solve $x+5\sqrt{37-x}=43$.

Process: Subtract 37 from each side of the equation, obtaining

$$x-37+5\sqrt{37-x}=6$$

which may be written

$$-(37-x)+5\sqrt{37-x}=6$$

or

$$(37-x)-5\sqrt{37-x}=-6.$$

Putting y for $\sqrt{37-x}$ this becomes

$$y^2-5y=-6.$$

Solving, $\qquad y=3$ or 2.

That is $\qquad \sqrt{37-x}=3$ or 2

whence $\qquad 37-x=9$ or 4

and $\qquad x=28$ or 33.

The same example may be treated by the method of Art. 11.

7. Solve $x^2-\sqrt{x^2-9}=21$.

8. Solve $2\sqrt{x^2-5x+2}-x^2+8x=3x-78$.

9. Solve $(2x^2-3x+1)^2=22x^2-33x+11$.

10. Solve $4x^2-4x+20\sqrt{2x^2-5x+6}=6x+66$.

The following are examples of either the theorem of Art. 16 or of Art. 17:

11. Solve $ax\sqrt[3]{x}+\frac{bx}{\sqrt[3]{x}}=c$.

12. Solve $x^{-2}-2x^{-1}=8$.

13. Solve $x^{\frac{2}{n}}-5x^{\frac{1}{n}}+4=0$.

14. Solve $3x-20=7\sqrt{x}$.

15. Solve $10x^{\frac{1}{2n}}+x^{\frac{1}{n}}+24=0$.

16. Solve $110x^{-4}+1=21x^{-2}$.

17. Solve $\sqrt{x}+4x^{-\frac{1}{2}}=5$.

18. Solve $3x^2-4x+\sqrt{3x^2-4x-6}=18$.

19. Prove that the equation $x^8-97x^4+1500=204$ is equivalent to the equation $(x^4-16)(x^4-81)=0$.

20. Solve $2x^{\frac{1}{3}}-3x^{\frac{2}{3}}+x^{\frac{3}{3}}=0$.

Result: $x=0$, or 1, or 8.

21. Solve $(x-a)^n+\frac{1}{(x-a)^n}=r$.

22. Solve $8x^{\frac{3}{2n}}-8x^{-\frac{3}{2n}}=63$.

23. Solve $x^{\frac{1}{5}}+8x^{\frac{7}{5}}+9x^{\frac{4}{5}}=0$.

CHAPTER VII.

SYSTEMS OF EQUATIONS.

1. Definition. If a number of equations containing several unknown quantities are supposed to be so related that they are all satisfied simultaneously by the *same set of values* of the unknown quantities, the equations are said to constitute a *System*, or a *System of Simultaneous Equations*.

Thus the equations

$$\left.\begin{aligned} 2x+y+5z&=19 \\ 3x+2y+4z&=19 \\ 7x+4y+12z&=49 \end{aligned}\right\}$$

are satisfied simultaneously by the set of values,

$$x=1,\ y=2,\ z=3,$$

and are said to constitute a system. This set of values, or the process of finding them, may be called the *Solution* of the system.

The reader is supposed to be already familiar with methods of solution of a system of simple equations containing as many equations as different unknown quantities, such as the system given above. The systems we propose to consider in this chapter are those involving quadratics or equations of higher degrees.

2. The student should not suppose that every system of equations which may be proposed is capable of solution. It is one requirement that the number of unknown quantities be just equal to the number of equations in the system. But even this is not all. Some of the equations in the system may contradict some of the others, in which case a solution is impossible. For example, take the system

$$\left.\begin{aligned} 3+2y&=2x \qquad (1) \\ x-y&=1 \qquad (2) \end{aligned}\right\}$$

From equation *(2)*

$$x=1+y.$$

Substitute this value of x in equation *(1)* and we obtain

$$3+2y=2+2y,$$

or

$$1=0,$$

and by no other method of elimination can we get anything but an absurdity from the given system. Equations of this kind are said to be *incompatible* because one equation affirms what another denies. We will see this to be so in the above system if, by proper transformations in equation (1), the system be written

$$\left.\begin{array}{ll} x-y=1\tfrac{1}{2} & (3) \\ x-y=1 & (4) \end{array}\right\}$$

These equations are necessarily contradictory and can have no solution.

Another example of an incompatible system is

$$\left.\begin{array}{r} x^2-y^2=4z \\ x-y=4 \\ x+y-z=2 \end{array}\right\}$$

From the second of these equations it is seen that

$$x=4+y.$$

Substituting this value of x in the first and third of the equations in order to eliminate x, we obtain the system

$$\left.\begin{array}{r} 2y-z=-4 \\ 2y-z=-2 \end{array}\right\}$$

and, since these are incompatible, we can go no further.

There is still another case in which a system may have no solution. Consider the equations

$$\left.\begin{array}{r} \tfrac{2}{3}x+\tfrac{3}{2}y=1 \\ 4x=3(2-3y) \end{array}\right\}$$

From the first equation we find

$$x=\tfrac{3}{2}-\tfrac{9}{4}y.$$

Substituting this value of x in the second equation we obtain

$$6-9y=3(2-3y),$$

which reduces to $0=0$,

and we get no solution. Equations of this kind are said to be *dependent* because the equations really make the *same statement* about the unknown quantities. This will be seen when, by proper transformations in the equations, the above system is written.

$$\left.\begin{array}{r} 4x+9y=6 \\ 4x+9y=6 \end{array}\right\}$$

It is now seen that the equtions of the system do not state independent truths, and consequently the system has no more meaning than a single equation containing two unknown quantities.

It will also be found that the system

$$\left.\begin{array}{r} x+2y+3z=-4 \\ 4x+3y+2z=1 \\ x+3(z+2)=2(1-y) \end{array}\right\}$$

is a dependent one, the dependence being between the first and third equations.

We may then enumerate three conditions which must be fulfilled by a system of equations in order that a solution may exist:

There must be just as many equations as there are unknown quantities.

The equations must be compatible.

The equations must be independent.

3. Of course if any equation of a system be operated upon in any manner during the solution, care must be taken that the transformation be with a due regard to the theorems in VI, Arts. 3—12. Obviously, no operation which it is questionable to perform on an equation standing alone can be legitimately performed upon one belonging to a system. But in addition to the reductions which single equations may undergo, equations of a system permit of certain transformations peculiar to themselves, and it remains to investigate the possible effect of these on the solution of the system. The following theorems are designed to point out the effect on the result of the ordinary steps in the process of elimination.

4. THEOREM. *If from the system of equations*

$$\left.\begin{array}{c} L_1=R_1 \\ L_2=R_2 \\ \cdot\ \cdot\ \cdot\ \cdot \\ L_n=R_n \end{array}\right\} \quad (a)$$

we derive the system

$$\left.\begin{array}{c} L_1=R_1 \\ L_1S+L_2T=R_1S+R_2T \\ \cdot\ \cdot\ \cdot\ \cdot \\ L_n=R_n \end{array}\right\} \quad (b)$$

where all but the second equation remain unchanged, the derivation is legitimate if T is a known quantity, not zero, but questionable if T is a function of the unknown quantities, it being indifferent whether S is a known quantity or a function of the unknown ones.

Write system *(a)* so that it will read

$$\left.\begin{array}{lr} L_1-R_1=0 & (1) \\ L_2-R_2=0 & (2) \\ \cdots\cdots & \\ L_n-R_n=0 & \end{array}\right\} (c)$$

and system *(b)* so that it will appear

$$\left.\begin{array}{lr} L_1-R_1=0 & (3) \\ S(L_1-R_1)-T(L_2-R_2)=0 & (4) \\ \cdots\cdots & \\ L_n-R_n=0 & \end{array}\right\} (d)$$

First, suppose T a known quantity.

Then any set of values that will satisfy *(c)* must make L_1-R_1, L_2-R_2, . . . and L_n-R_n each zero. But any set that makes these zero must satisfy *(d)* also. Hence any solution of *(c)* is a solution of *(d)*.

It is seen from *(3)* that any set of values that satisfies *(d)* must make L_1-R_1 zero. Equation *(4)* will then become

$$T(L_2-R_2)=0. \qquad (5)$$

Now since T is a known quantity, not zero, this cannot be satisfied unless L_2-R_2 is zero. Hence any set of values, in order to satisfy *(d)*, must make L_1-R_1 and L_2-R_2 and also . . . L_n-R_n each zero. But any set of values that makes these zero will satisfy *(c)*. Therefore any solution of *(d)* is a solution of *(c)*.

Now we have shown, *first*, that any set of values that will satisfy *(c)* will satisfy *(d)*, and *second*, that any set of values that will satisfy *(d)* will satisfy *(c)*. Hence the two systems are equivalent.

Second, suppose T a function of some of the unknown quantities.

In this case equation *(5)* may be satisfied by any set of values that will satisfy the equation

$$T=0$$

without assuming that L_2-R_2 is zero. Consequently *(d)* can be satisfied without equation *(2)* being satisfied; that is, without *(c)*

being satisfied. Therefore *(d)* is not equivalent to *(c)* but to the two systems

$$\left.\begin{array}{r} L_1-R_1=0 \\ L_2-R_2=0 \\ \cdots\cdots \\ L_n-R_n=0 \end{array}\right\}$$

$$\left.\begin{array}{r} L_1-R_1=0 \\ T=0 \\ \cdots\cdots \\ L_n-R_n=0 \end{array}\right\}$$

5. EXAMPLES. The derivation discussed in the above theorem is the one so frequently used in elimination. Thus take the system

$$\left.\begin{array}{rl} 2x+y=17 & (1) \\ 5x-10y=\ 5 & (2) \end{array}\right\}$$

Multiply *(1)* through by 5 and *(2)* through by 2 and obtain a new equation by subtracting the former from the latter and the system becomes

$$\left.\begin{array}{rl} 2x+y=17 & (3) \\ 25y=75 & (4) \end{array}\right\}$$

We have eliminated x from the second equation and consequently y is readily found to equal 3.

From *(3)*, x is then found to equal 7.

The theorem shows it is also legitimate to transform

$$\left.\begin{array}{r} x+3=y \\ x^2+6=xy \end{array}\right\}$$

into

$$\left.\begin{array}{r} x+3=y \\ 6-3x=0 \end{array}\right\}$$

by multiplying the first equation through by x and subtracting the resulting equation from the second.

An example of the use of the following theorem will be found in V, Art. 12 *(c)*.

6. THEOREM. *It is legitimate to derive from the system*

$$\left.\begin{array}{r} L_1=R_1 \\ L_2=R_2 \end{array}\right\} \qquad (a)$$

the system

$$\left.\begin{array}{r} L_1=R_1 \\ SL_1^2+TL_2=SR_1^2+TR_2 \end{array}\right\} \qquad (b)$$

if T is a known quantity, not zero.

Rewrite *(a)* and *(b)* so that they shall read

$$\left.\begin{aligned} L_1-R_1&=0 \qquad (1) \\ L_2-R_2&=0 \qquad (2) \end{aligned}\right\} \qquad (c)$$

and

$$\left.\begin{aligned} L_1-R_1&=0 \qquad (3) \\ S(L_1^2-R_1^2)+T(L_2-R_2)&=0 \qquad (4) \end{aligned}\right\} \qquad (d)$$

It is evident that any set of values which will satisfy *(c)* will satisfy *(d)*, for whatever makes L_1-R_1 and L_2-R_2 each zero will satisfy *(d)*.

It is seen from *(3)* that any set of values that satisfies *(d)* must make L_1-R_1 zero. Equation *(4)* will then become

$$T(L_2-R_2)=0. \qquad (5)$$

Now, since T is a known quantity not zero, this cannot be satisfied unless L_2-R_2 is zero. Hence any set of values that satisfies *(d)* must make L_1-R_1 and L_2-R_2 each zero; that is, must be a solution of *(c)*.

Now, since any solution of *(c)* is a solution of *(d)* and any solution of *(d)* is a solution of *(c)*, the two systems are equivalent.

7. THEOREM. *If from a system containing two unknown quantities*

$$\left.\begin{aligned} L_1&=R_1 \qquad (1) \\ L_2&=R_2 \qquad (2) \end{aligned}\right\} \qquad (a)$$

we derive the system

$$\left.\begin{aligned} L_1&=R_1 \qquad (3) \\ L_1L_2&=R_1R_2 \qquad (4) \end{aligned}\right\} \qquad (b)$$

the derivation is questionable if L_1 *and* R_1 *both involve unknown quantities, but legitimate if either is a known quantity not zero.*

First, suppose that L_1 and R_1 each involve unknown quantities.

Any value of the unknown quantities which will satisfy the equation

$$L_1=0$$

must satisfy equation *(4)*, since the relation $L_1=R_1$ must hold if system *(b)* is to be satisfied.

Also any value of the unknown quantities which will satisfy the equation

$$R_1=0$$

must satisfy *(4)*, since the relation $L_1=R_1$ must hold if the system is to be satisfied.

Moreover, any value of the unknown quantities which will satisfy

$$L_2=R_2$$

must satisfy *(4)* since the relation $L_1=R_1$ must hold.

Therefore, from these considerations, it is evident that the system *(b)* is not equivalent to system *(a)*, but to the three systems

$$\left.\begin{matrix} L_1=R_1 \\ L_1=0 \end{matrix}\right\} \qquad (b_1)$$

$$\left.\begin{matrix} L_1=R_1 \\ R_1=0 \end{matrix}\right\} \qquad (b_2)$$

$$\left.\begin{matrix} L_1=R_1 \\ L_2=R_2 \end{matrix}\right\} \qquad (b_3)$$

Second, suppose that either L_1 or R_1 is a known quantity not zero.

One of them, say R_1, is the known quantity. Therefore L_1 cannot be zero, since the relation $L_1=R_1$ must hold. Hence the introduced system (b_2) and (b_3) are absurdities, since they require that L_1 and R_1 be zero. Consequently the derivation is legitimate since the introduced systems are incompatible.

8. EXAMPLES. As an illustration of the theorem, consider the system

$$\left.\begin{matrix} x-4=6-y \\ 2x+y=13 \end{matrix}\right\}$$

This is satisfied by $x=3$ and $y=7$. Now form the system

$$\left.\begin{matrix} x-4=6-y \\ (x-4)(2x+y)=13(6-y) \end{matrix}\right\}$$

which is satisfied by either of the sets of values, $x=3$, $y=7$ or $x=4$, $y=6$. The additional solution may be obtained from either of the systems

$$\left.\begin{matrix} x-4=6-y \\ x-4=0 \end{matrix}\right\}$$

$$\left.\begin{matrix} x-4=6-y \\ 6-y=0 \end{matrix}\right\}$$

As another example consider the system

$$\left.\begin{matrix} x-2y=3 \\ x+2y=7 \end{matrix}\right\}$$

which is satisfied by $x=5$, $y=1$. From this we may obtain the system

$$\left.\begin{aligned} x-2y&=3\\ x^2-4y^2&=21 \end{aligned}\right\}$$

From the first equation of the system

$$x=3+2y.$$

Substituting this value for x in the second equation, it becomes

$$9+12y+4y^2-4y^2=21;$$

whence $$y=1.$$

Therefore, from the first equation of the system,

$$x=5.$$

In this case we see that no solution has been introduced. In fact, the introduced systems become

$$\left.\begin{aligned} x-2y&=3\\ x-2y&=0 \end{aligned}\right\}$$

$$\left.\begin{aligned} x-2y&=3\\ 7&=0 \end{aligned}\right\}$$

which are incompatible.

9. THEOREM. *If from the system*

$$\left.\begin{aligned} L_1&=R_1\\ L_1L_2&=R_1R_2 \end{aligned}\right\} \qquad (a)$$

we derive the system

$$\left.\begin{aligned} L_1&=R_1\\ L_2&=R_2 \end{aligned}\right\} \qquad (b)$$

the derivation is questionaale if both L_1 and R_1 involve unknown quantities, but legitimate if either is a known quantity not zero.

First, suppose that L_1 and R_1 both involve unknown quantities.

Then, by Art. 7, if we pass from *(b)* to *(a)* we gain solutions. Hence to pass from *(a)* to *(b)* is to *lose* those solutions.

Second, suppose that either L_1 or R_1 is a known quantity.

Then, by Art. 7, if we pass from *(b)* to *(a)* no solutions are gained. Hence none are lost if we pass from *(a)* to *(b)*.

10. EXAMPLES. According to the above theorem it is legitimate to divide one equation by another, member by member, if one member is a known quantity *not zero*.

Thus take the system

$$\left.\begin{array}{r} x+y=5 \\ x^2-y^2=15 \end{array}\right\} \qquad (a)$$

and derive the system

$$\left.\begin{array}{r} x+y=5 \\ x-y=3 \end{array}\right\} \qquad (b)$$

The only set of values which will satisfy (a) is $x=4$, $y=1$. This set satisfies (b) and no solution is lost.

The system (a) is equivalent to the system (b) and to two other systems (see Art. 7), but the other two systems are incompatible.

As an example of the case in which solutions may be lost, consider the system

$$\left.\begin{array}{r} x=3-y \\ x^2=9-y^2 \end{array}\right\} \qquad (c)$$

which is satisfied by either of the two sets $x=0$, $y=3$ and $x=3$, $y=0$. If we divide the second equation by the first, member by member, we pass to the system

$$\left.\begin{array}{r} x=3-y \\ x=3+y \end{array}\right\} \qquad (d)$$

which is satisfied only by the values $x=3$, $y=0$.

11. SOLUTION OF A LINEAR-QUADRATIC SYSTEM. We now propose to take up the solution of those systems involving two unknowns which consist of one linear and one quadratic equation. It is convenient to call this a *linear-quadratic* system. We will proceed by first working the following particular example:

$$\left.\begin{array}{rl} x+y=5 & (1) \\ x^2-2y^2=1 & (2) \end{array}\right\} \qquad (a)$$

From equation (1) the value of x in terms of y is easily seen to be

$$x=5-y. \qquad (3)$$

Substituting this value of x in equation (2) we obtain

$$(5-y)^2-2y^2=1 \qquad (4)$$

or

$$25-10y+y^2-2y^2=1 \qquad (5)$$

Uniting and transposing terms

$$y^2+10y=24, \qquad (6)$$

whence, solving this quadratic,

$$y=2 \text{ or } -12,$$

and from equation (1)

$$x=3 \text{ or } 17.$$

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Consequently there are *two* sets of values which will satisfy system *(a)*, namely,

$$x=3,\ y=2,$$

and $$x=17,\ y=-12.$$

Now the method here used may be applied to the solution of any linear-quadratic system containing two unknowns. In fact, take the general case*

$$\left.\begin{array}{r} x+ay=b \\ x^2+cy^2+dxy+ex+fy=g \end{array}\right\} \qquad (b)$$

where a, b, c, d, e, f, and g are supposed to stand for any real quantities whatever.

The value of x in terms of y from the first equation of the system is

$$x=b-ay \qquad (7)$$

Substituting this for x in the second equation of the system, that equation becomes

$$b^2-2aby+a^2y^2+cy^2+bdy-ady^2+eb-aey+fy=g.$$

Combining together those terms which contain y^2 and those which contain y and transposing all the known terms to the right hand side of the equation, this becomes

$$(a^2+c-ad)y^2+(bd-2ab-ae+f)y=g-b^2-eb,$$

which is a quadratic in which y is the only unknown quantity, whence it can be solved. The values which may be found from this can be substituted in equation *(7)* above, and the values of x will be determined.

12. Examples. Solve the following systems:

1. $\begin{cases} x+y=7 \\ x^2+2y^2=74. \end{cases}$

2. $\begin{cases} x+y=5 \\ 2x+xy+2y=16. \end{cases}$

3. $\begin{cases} (x-5)(y+2)=10 \\ 4x=3y. \end{cases}$

4. $\begin{cases} x+y=4 \\ xy=96. \end{cases}$

*It might be thought that this is not a general case, since x in the first equation and x^2 in the second do not appear with coefficients. But if either of them had a coefficient the equation could be reduced to the given form by dividing through by that very coefficient.

5. $\begin{cases} x-\dfrac{x-y}{2}=4 \\ \dfrac{1}{x}-\dfrac{2}{x-y}=1. \end{cases}$

13. SOLUTION OF SYSTEMS OF TWO QUADRATICS. If we have a system of two quadratic equations containing two unknown quantities and attempt to eliminate one of the unknown quantities it will be found in general that the resulting equation is of the *fourth degree.* Thus take the system

$$\left.\begin{array}{l} x^2-y=5 \\ y^2+xy=10. \end{array}\right\}$$

We find from the first equation that

$$y=x^2-5.$$

Substitute this value for y in the second equation, and it becomes

$$(x^2-5)^2+x(x^2-5)=10$$

or, expanding and collecting terms,

$$x^4+x^3-5x^2-5x-25=10.$$

Now, since we are not yet familiar with the solution of equations of a degree higher than the second, the treatment of systems of two quadratics in general cannot be taken up at this place. But there are two important special cases of systems of two quadratics whose treatment will involve no knowledge beyond the solution of quadratic equations, and these we will now consider. The cases referred to are

I. Where the terms in each equation containing the unknown quantities constitute a homogeneous expression with respect to the unknown quantities.

II. Where the equations are symmetrical.*

14. CASE I. We will illustrate the first case, and also the method of elimination which may be applied to any example of it, by the following solution:

Solve the system $\begin{cases} x^2-2xy=5 & (1) \\ 3x^2-10y^2=35. & (2) \end{cases}$

*For the definitions of homogeneous and symmetrical see I, Arts. 7 and 8.

Suppose $x=vy$, where v is a new unknown quantity. Then, substituting this in the equations, the system becomes

$$\left.\begin{array}{ll} v^2y^2-2vy^2=5 & (3) \\ 3v^2y-10y^2=35. & (4) \end{array}\right\} \qquad (a)$$

From equation (3) we find that

$$y^2=\frac{5}{v^2-2v} \qquad (5)$$

and from equation (4)

$$y^2=\frac{33}{3v^2-16.} \qquad (6)$$

Whence, from (5) and (6)

$$\frac{5}{v^2-2v}=\frac{35}{3v^2-10.}$$

Clearing of fractions,

$$15v^2-50=35v^2-70v.$$

Transposing and uniting terms, and dividing through by 10

$$2v^2-7v=-5.$$

Solving this quadratic

$$v=\tfrac{5}{2} \text{ or } 1.$$

Substituting the first of these values in (3) we obtoin

$$\tfrac{25}{4}y^2-5y^2=5.$$

Whence $\qquad y=\pm 2$

and since $\qquad x=vy$

$$x=\pm 5.$$

Now, substituting the second value of v in (3) we obtain

$$y^2-2y^2=5.$$

Whence $\qquad y=\pm\sqrt{-5}$

and, since $x=vy$,

$$x=\pm\sqrt{-5}.$$

Therefore we have, as the solution of the original system, the four sets of values.

$$x=5,\ y=2;\quad x=-5,\ y=-2;\quad x=\sqrt{-5},\ y=\sqrt{-5};$$
$$x=-\sqrt{-5},\ y=-\sqrt{-5}.$$

The general system of two equations of the above class may be represented by

$$\left.\begin{array}{l} x^2+axy+by^2=c \\ x^2+dxy+ey^2=f. \end{array}\right\} \qquad (b)$$

The student may show that the method set forth above will solve the general system and hence any possible example under it.

15. EXAMPLES. Solve the following systems:

1. $\begin{cases} 3xy-y^2=18 \\ 2x^2+y^2=54. \end{cases}$

2. $\begin{cases} \dfrac{x}{y}+\dfrac{y}{x}=2\frac{1}{2} \\ \dfrac{x^2-y^2}{xy}=1\frac{1}{2}. \end{cases}$

3. $\begin{cases} x^2+xy=15 \\ xy-y^2=2. \end{cases}$

4. $\begin{cases} x^2-y^2=40 \\ xy=11. \end{cases}$

5. $\begin{cases} \dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}=\dfrac{10}{3} \\ x^2+y^2=45. \end{cases}$

16. CASE II. To show that any system of two symmetrical quadratics can be solved, we will start with the general case, which is evidently

$$\left.\begin{aligned} x^2+ax+bxy+ay+y^2=c \\ x^2+dx+exy+dy+y^2=f \end{aligned}\right\} \qquad (a)$$

If x^2 and y^2 appeared in either of these equations with coefficients the system could be reduced to the given form by dividing the equation through by that coefficient.

Through the given system substitute $u+w$ for x and $u-w$ for y, where u and w are two new unknown quantities. Then *(a)* becomes

$$\left.\begin{aligned} 2u^2+2w^2+2au+bu^2-bw^2=c \\ 2u^2+2w^2+2du+eu^2-ew^2=f \end{aligned}\right\} \qquad (b)$$

Subtracting the second of these equations from the first we obtain

$$2(a-d)u+(b-e)u^2-(b-e)w^2=c-f,$$

or

$$2(a-d)u+(b-e)(u^2-w^2)=c-f.$$

Whence

$$u=\frac{c-f-(b-e)(u^2-w^2)}{2(a-d)}$$

Now if the right-hand side of this equation be substituted for u in the terms $2au$ and $2du$ of system *(b)*, that system will contain no powers of the unknown quantities but the second and will

therefore come under Case I. When u and w are thus found, x and y can be determined from the equations.

$$x=u+w$$
$$y=u-w.$$

The above work shows that Case II can always be solved, but we do not pretend that the method used is always the most economical one to employ. The insight and ingenuity of the student will often suggest special expedients for particular examples which are preferable to a general method.

17. EXAMPLES. Solve the following systems:

1. $$\begin{cases} x+xy+y=65 \\ xy \quad =50. \end{cases}$$

2. $$\begin{cases} 4(x+y)=3xy \\ x^2+x+y+y^2=26. \end{cases}$$

3. $$\begin{cases} \frac{1}{x}+\frac{1}{y}=\frac{14}{45} \\ xy=1. \end{cases}$$

4. $$\begin{cases} x^2+y^2=18\frac{1}{4} \\ xy=6. \end{cases}$$

A common expedient for readily solving such a system is to first transform it into the system

$$\begin{cases} x^2-2xy+y^2=6\frac{1}{4} \\ x^2+2xy+y^2=30\frac{1}{4} \end{cases}$$

from which the values of $x-y$ and $x+y$ can be found and consequently the values of x and y.

5. $$\begin{cases} \frac{x^2+y^2}{x+y}=\frac{85}{9} \\ \frac{x+y}{xy}=\frac{18}{77} \end{cases}$$

18. MISCELLANEOUS SYSTEMS. We have enumerated all the classes of systems involving equations of a degree higher than the first which can invariably be solved without a knowledge of the solution of cubic and higher equations. The solvable cases embrace but a small fraction of the systems which may arise. Of the numbers remaining a still smaller proportion can be solved by special expedients. The great mass of systems involving quadratic or higher equations are thus irreducible by straightforward methods of solution, *i. e.*, as a rule, such systems are insolvable, not-

withstanding a chance exception. In those systems which may be solved, special expedients are more to be sought for than general methods. In fact, sharp inspection of the equations and a knowledge of algebraic forms will often be the means of discovering an ingenious solution for an apparently insolvable system.

The theorems of this chapter will be found useful either in justifying or in throwing doubt upon many of the common transformations during the ordinary solution of a system. As far as possible the student should endeavor to take account of all questionable derivations at the time they are made and make allowance for them in the result. In this connection the remarks at the beginning of the last chapter should not be forgotten. No matter how skillfully or ingeniously a set of values may have been obtained, *they must satisfy the original system, or it is no solution.* Whenever any derivation not allowed by the theorems is used, however plausible it may seem, this ultimate test must be applied.

The following systems include examples of the cases already considered, besides others requiring special treatment. The method of Case II will be found to solve many symmetrical systems of high degree.

1. $\begin{cases} x+y=5 & (1) \\ x^5+y^5=275. & (2) \end{cases}$

Put $x=u+w$ and $y=u-w$, whence the system becomes

$$\begin{cases} 2u=5 \\ 2u^5+20u^3w^2+10uw^4=275. \end{cases}$$

From the first of these equations $u=\frac{5}{2}$. Whence, substituting this in the second, we obtain

$$\frac{3125}{16}+\frac{625}{2}w^2+\frac{125}{2}w^4=275$$

which is a quadratic in terms of w^2. When w is found x and y can be found from the equations $x=u+w$ and $y=u-w$.

2. $\begin{cases} x^2y^2=10xy+9000 \\ x^2+y^2=200. \end{cases}$

3. $\begin{cases} x-y=2 \\ x^3-y^3=8. \end{cases}$

4. $\begin{cases} x+y=5 \\ x^3+y^3=65. \end{cases}$

5. $\begin{cases} x^2+3xy=54 \\ xy+4y^2=115. \end{cases}$

6. $\begin{cases} x^2+y^2=a \\ xy=b. \end{cases}$

7. $\begin{cases} x^2+xy+2y^2=2 \\ 2x^2+xy+y^2=2. \end{cases}$

8. $\begin{cases} x^2+5xy+y^2=51 \\ x^2-xy+y^2=21. \end{cases}$

9. $\begin{cases} x+y=\sqrt{5} \\ x^4+y^4=7x^2y^2. \end{cases}$

10. $\begin{cases} x^2y+y^2x=20 \\ \frac{1}{x}+\frac{1}{y}=\frac{5}{4} \end{cases}$

11. $\begin{cases} (x+y)xy=240 \\ x^3+y^3=280. \end{cases}$

12. $\begin{cases} x^2+y^2+x+y=18 \\ x^2-y^2+x-y=6. \end{cases}$

13. $\begin{cases} (x+5)(y-2)=0 \\ (x-4)(y-7)=0. \end{cases}$

14. $\begin{cases} x^2+y^2-12=x+y \\ xy+8=2(x+y) \end{cases}$

15. $\begin{cases} x^4-x^2+y^4-y^2=84 \\ x^2+x^2y^2+y^2=49. \end{cases}$

16. $\begin{cases} \frac{10x+y}{xy}=3 \\ 9(y-x)=18 \end{cases}$ Result, $x=4$ or $\frac{5}{3}$. $y=2$ or $-\frac{1}{2}$.

17. $\begin{cases} a(x-y)=b(x+y) \\ x^2-y^2=c^2. \end{cases}$

18. $\begin{cases} x^2-xy=2 \\ 2x^2+y^2=9 \end{cases}$ Result, $x=\pm\frac{1}{\sqrt{3}}$; $y=\pm\frac{5}{\sqrt{3}}$.

19. $\begin{cases} x^2+y^2-x-y=78 \\ xy+x+y=30. \end{cases}$ Result, $x=9$ or $-\frac{13}{2}+\frac{1}{2}\sqrt{-39}$. $y=3$ or $-\frac{13}{2}-\frac{1}{2}\sqrt{-39}$.

20. $\begin{cases} x^2+y^2=34 \\ 2x^2-3xy=23-2y^2. \end{cases}$ Result, $x=\pm5$, or ±3. $y=\pm3$, or ±5.

21. $\begin{cases} \sqrt{x}+\sqrt{y}=3 \\ x^{\frac{3}{2}}+y^{\frac{3}{2}}=9. \end{cases}$ Result, $x=4$. $y=1$.

22. $\begin{cases} \sqrt{x}-\sqrt{y}=2 \\ (x+y)\sqrt{xy}=510. \end{cases}$ Result, $x=5$, or -3. $y=3$, or -5.

23. $\begin{cases} xy = 1225 \\ \sqrt{x} + \sqrt{y} = 12. \end{cases}$ Result, $x = 49$ or 25.
$y = 25$ or 49.

24. $\begin{cases} x + y = a - \sqrt{x+y} \\ x - y = b. \end{cases}$

Result, $x = \dfrac{2(a+b) + 1 \pm \sqrt{4a+1}}{4}$

$y = \dfrac{2(a-b) + 1 \pm \sqrt{4a+1}}{4}$

CHAPTER VIII.

PROGRESSIONS.

1. DEFINITIONS. An *Arithmetical Progression* is a series of terms such that each differs from the preceding by a fixed quantity, called the *common difference.* The following are examples:

$$7+9+11+13+15+\dots$$
$$31+26+21+16+11+\dots$$
$$a+(a+d)+(a+2d)+(a+3d)+\dots$$
$$(x-y)+x+(x+y)+\dots$$
$$(x-3y)+(x-y)+(x+y)+(x+3y)+\dots$$

The first and last terms of any given progression are called the *Extremes*, and the other terms the *Means*.

2. TO FIND THE nTH TERM OF AN ARITHMETICAL PROGRESSION. Represent the first term of the progression by a and the common difference by d. Then we have

Number of term.	1.	2.	3.	4.	5.
Progression.	a +	$(a+d)$ +	$(a+2d)$ +	$(a+3d)$ +	$(a+4d)$, etc.

We notice that by the nature of the progression every time the number of terms is increased by 1 the coefficient of d is increased by 1 also; hence to get the nth term from the 5th term, the common difference must be added to it $n-5$ times. Whence, representing the nth term by l, $l=a+4d+(n-5)d$, or

$$l=a+(n-1)d. \qquad (1)$$

3. TO FIND THE SUM OF n TERMS OF AN ARITHMETICAL PROGRESSION. Representing the sum of the arithmetical progression by s, we have

$$s=a+(a+d)+(a+2d)+(a+3d)+\dots+l, \qquad (1)$$

or, writing this progression in reverse order, we have

$$s=l+(l-d)+(l-2d)+(l-3d)+\dots+a \qquad (2)$$

Now adding (1) and (2) together term for term, noticing that the common difference vanishes, we have

$$2s=(a+l)+(a+l)+(a+l)+(a+l)+\dots+(a+l).$$

If the number of terms in the original progression be called n, this becomes

$$2s=n(a+l),$$

whence
$$s=\tfrac{1}{2}n(a+l).$$

4. To Insert any Number of Arithmetical Means between two given Quantities. Suppose we are to insert p arithmetical means between the two terms a and l. The whole number of terms in the progression consists of the r means and the two extremes. Hence the number of terms in the progression is $p+2$. Therefore, substituting in (1), Art. 2, we obtain

$$l=a+(p+2-1)d,$$
$$d=\frac{l-a}{p+1},$$

and now, since the common difference is known, any number of means can be found by repeated additions.

5. The two equations

$$\begin{cases} l=a+(n-1)d & (1) \\ s=\frac{1}{2}n(a+l) & (2) \end{cases}$$

contain five different quantities. If any two of them are unknown and the values of the rest are given the values of the two unknown can be determined by a solution of the system. As an example, suppose that a and d are unknown and the rest known. Putting x for a and y for d so that the unknown quantities will appear in their usual form, the system becomes

$$\begin{cases} l=x+(n-1)y & (3) \\ s=\frac{1}{2}n(x+l) & (4) \end{cases}$$

Finding the value of x in each equation the system becomes

$$\begin{cases} x=l-(n-1)y & (5) \\ x=\frac{2s}{n}-l, & (6) \end{cases}$$

Whence, equating right-hand members of (5) and (6), we obtain

$$l-(n-1)y=\frac{2s}{n}-l,$$

whence
$$y=\frac{2nl-2s}{n(n-1)} \qquad (7)$$

Therefore, restoring a and d in (6) and (7),

$$a=\frac{2s}{n}-l,$$

$$d=\frac{2nl-2s}{n(n-1)}.$$

where a and d are expressed in terms of the three known quantities, n, l, snd s. In like manner we may suppose *any* two of the five quantities unknown and find their values in terms of the known ones. In all, ten different cases may arise, which are given in the following table, each of which is to be worked by the student.

No	*Given.*	*Required*	*Result.*	
1.	$a, d, n,$	$l, s,$	$l=a+(n-1)d$;	$s=\frac{1}{2}n[2a+(n-1)d]$.
2.	$l, d, n,$	$a, s,$	$a=l-(n-1)d$;	$s=\frac{1}{2}n[2l-(n-1)d]$.
3.	$a, l, n,$	$d, s,$	$d=\frac{l-a}{n-1}$;	$s=\frac{1}{2}n(a+l)$.
4.	$a, n, s,$	$d, l,$	$d=\frac{2s-2an}{n(n-1)}$;	$l=\frac{2s}{n}-a$.
5.	$n, d, s,$	$a, l,$	$a=\frac{s}{n}-\frac{(n-1)d}{2}$;	$l=\frac{s}{n}+\frac{2nl-2s}{n(n-1)}$.
6.	$l, n, s,$	$a, d,$	$a=\frac{2s}{n}-l$;	$d=\frac{2nl-2s}{n(n-1)}$.
7.	$a, d, l,$	$n, s,$	$n=\frac{l-a}{d}+1$;	$s=\frac{(l-a)(l-a+d)}{2d}$.
8.	$a, l, s,$	$n, d,$	$n=\frac{2s}{a+l}$;	$d=\frac{l^2-a^2}{2s-a-l}$.
9.	$a, d, s,$	$l, n,$	$l=-\frac{1}{2}d\pm\sqrt{2ds+(a-\frac{1}{2}d)^2}$;	$n=\frac{d-2a\pm\sqrt{(2a-d)^2+8ds}}{2d}$
10.	$l, d, s,$	$a, n,$	$a=\frac{1}{2}d\pm\sqrt{(l+\frac{1}{2}d)^2-2ds}$;	$n=\frac{2l+d\pm\sqrt{(2l+d)^2-8ds}}{2d}$

6. Examples.

1. Find the sum of 9 terms of the progsession $3+7+11$, etc.

2. The first term is 96, the common difference -5; what is the 13th term?

3. The first term is $8\frac{1}{8}$, the common difference $-\frac{5}{8}$, and the number of terms 29; what is the sum?

4. The first term is $\frac{3}{4}$, the common difference $\frac{3}{4}$, and the number of terms 12; what is the sum?

5. Insert 10 arithmetical maeans between $-\frac{7}{8}$ and $+\frac{7}{8}$.

6. Find the sum of the first n odd numbers $1+3+5+7$, etc.

7. Find the sum of n terms of the progression of natural numbers $1+2+3+4$, etc.

8. Find the sum of n terms of the progression of even numbers $0+2+4+6+8$, etc.

9. The first term is 11, the common difference -2, and the sum 27. Find the number of terms.

10. The first term is 4, the common difference is 2, and the sum 18. Find the number of terms.

11. The first term is 11, the common difference is -3, and the sum 24. Find the number of terms.

12. The sum of n consecutive odd numbers is s. Find the first of the numbers.

13. Select 10 consecutive numbers from the natural scale whose sum shall be 1000.

14. Sum $\sqrt{\frac{1}{2}}+\sqrt{2}+3\sqrt{\frac{1}{2}}+$ etc., to twenty terms.

15. Sum $5-2-9-16-$ etc., to eight terms.

16. Find the tenth term of the arithmetical progression whose first and sixteenth terms are 2 and 48; and also determine the sum of those eight terms the last of which is 60.

17. Insert five arithmetical means between 10 and 8.

18. Insert four arithmetical means between -2 and -16.

19. How many terms must be taken from the commencement of the series $1+5+9+13+17$ etc., so that the sum of the 13 succeeding terms may be 741?

20. Wnat is the expression for the sum of n terms of an arithmetical progression whose first term is $\frac{3}{2}$ and the difference of whose third and seventh terms is 3?

21. The sum of the first three terms of an arithmetical pro-

gression is 15 and the sum of their squares is 83; find the common difference.

22. There are two arithmetrical series which have the same common difference; the first terms are 3 and 5 respectively and the sum of seven terms of the one is to the sum of seven terms of the other as 2 to 3. Determine the series.

7. DEFINITIONS. *A Geometrical Progression* is a series of terms such that each is the product of the preceding by a fixed factor called the *Ratio*. The following are examples:

$$3+6+12+24+48, \text{ etc.}$$
$$100+50+25+12\tfrac{1}{2}+6\tfrac{1}{4}, \text{ etc.}$$
$$\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}+\tfrac{1}{32}, \text{ etc.}$$
$$\tfrac{1}{3}+\tfrac{1}{6}+\tfrac{1}{12}+\tfrac{1}{24}+\tfrac{1}{48}, \text{ etc.}$$

The first and last terms of any progression are often called the *Extremes* and the remaining terms the *Means*.

8. TO FIND THE nTH TERM. Let a represent the first term of the geometrical progression and r the ratio. Then the progression may be written:

Number of term. 1. 2. 3. 4. 5.
Progression. $a+ar+ar^2+ar^3+ar^4$.

We notice that, by the nature of the progression, every time the number of terms is increased 1 the exponent of r is increased by 1 also; hence to get the nth term from the 5th term it must be multiplied by the ratio $n-5$ times. Whence. reoresenting the nth term by l, and $l=(ar^4)(n-3)$,
or

$$l=ar^{n-1}. \qquad (1)$$

9. TO FIND THE SUM OF n TERMS. Represnting the sum of the geometrical progression by s we have

$$s=a+ar+ar^2+ar^3+ \ . \ . \ . \ +ar^{n-2}+ar^{n-1}. \qquad (1)$$

Multiplying this equation through by $r-1$, we obtain

$$(r-1)s=ar^n-a.$$

Whence

$$s=\frac{ar^n-a}{r-1} \qquad (2)$$

Now $ar^n=r(ar^{n-1})$. Therefore, since $ar^{n-1}=l$,

$$ar^n=al.$$

Whence, substituting this value of ar^n in (2) we obtain as another expression for s

$$s=\frac{rl-a}{r-1} \qquad . \qquad (3)$$

10. To Insert any Number of Geometrical Means Between two Given Quantities. Suppose we are to insert p geometrical means between the two terms a and l. The whole number of terms in the progression is therefore $p+2$. Hence, substituting $p+2$ for n in (1), Art. 8,

$$l=aa^{n-1}.$$

Consequently

$$r=\left\{\frac{l}{a}\right\}^{\frac{1}{p+1}},$$

and now, since the ratio is known, any number of means can be found by repeated multiplications.

11. The two equations

$$\begin{cases} l=ar^{n-1} \\ s=\dfrac{ar^n-a}{r-1} \end{cases}.$$

contain five different quantities. If any two of them are unknown, and the values of the rest are given, the values of the two unknown can be determined by a solution of the system. But if r is an unknown quantity the equations of the system are of a high degree, since n is usually a large number and always greater than 2 at least. In this case we will be unable to solve the system, as it is one beyond the range cf Chapter VII. Also if n is an unknown quantity, we will have an equation with the unknown quantity appearing as an *exponent*, which is a kind of equation we have not yet discussed. Hence there are a limited number of cases in which we can solve the above system. The following table contains the ten possible cases, with the solutions as far as possible. The values of n in the last four are printed merely to make the table complete, for the manner of obtaining them is not explained until Chapter XV is reached.

No	Given.	Required	Result.
1.	$a, r, n,$	$l, s,$	$l=ar^{n-1}$; $\quad s=\dfrac{ar^n-a}{r-1}$.
2.	$l, r, n,$	$a, s,$	$a=\dfrac{l}{r^{n-1}}$; $\quad s=\dfrac{lr^n-l}{r^n-r^{n-1}}$.
3.	$n, r, s,$	$a, l,$	$a=\dfrac{(r-1)s}{r^n-1}$; $\quad l=\dfrac{(r-1)sr^{n-1}}{r^n-1}$.
4.	$a, l, n,$	$r, s,$	$r=\left(\dfrac{l}{a}\right)^{\frac{1}{n-1}}$; $\quad s=\dfrac{l^{\frac{n}{n-1}}-a^{\frac{n}{n-1}}}{l^{\frac{1}{n-1}}-a^{\frac{1}{n-1}}}$
5.	$a, n. s,$	$r, l,$	$ar^n-rs=a-s$; $\quad l(s-l)^{n-1}=a(s-a)^{n-1}$.
6.	$l, n, s,$	$a, r,$	$a(s-a)^{n-1}=l(s-l)^{n-1}$; $\quad (s-l)r^n-sr^{n-1}=-l$.
7.	$a, r, l,$	$s, n,$	$s=\dfrac{lr-a}{r-1}$; $\quad n=\dfrac{\log l-\log a}{\log r}+1$.
8.	$a, l, s,$	$r, n,$	$r=\dfrac{s-a}{s-l}$; $\quad n=\dfrac{\log l-\log a}{\log (s-a)-\log (s-l)}+1$.
9.	$a, r, s,$	$l, n,$	$l=\dfrac{a+(r-1)s}{r}$; $\quad n=\dfrac{\log [a+(r-1)s]-\log a}{\log r}$.
10.	$l, r, s,$	$a, n,$	$a=lr-(r-1)s$; $\quad n=\dfrac{\log l-\log [lr-(r-1)s]}{\log r}+1$.

12. EXAMPLES AND PROBLEMS.

1. Find the sum of 10 terms of the progression $3+9+27+$etc.

2. Find the sum of 10 terms of the series $\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}$, etc., or the series $.333+$

3. Find the sum of 100 terms of the progression $.3333+$ etc.

4. Sum 5 terms of the progression $27+270+2700$, etc.

5. Sum 10 terms of the progression $4-2+1-$ etc.

6. Sum the series $\sqrt[6]{3}+\sqrt[6]{6}+\sqrt[6]{12}+$ etc., to eight terms.

7. Sum the series $3-2+\frac{4}{3}-\frac{8}{9}+$ etc., to nine terms.

8. Sum the series $-4+8-16+32$, etc., to 6 terms.

9. The fourth term of a geometrical progression is 192 and the seventh term is 12288; find the sum of the first three terms.

10. Prove that if quantities be in geometric progression their differences are also in geometrical progression, having the same common ratio as before.

11. The first and sixth terms of a geometric progression are 1 and 243; find the sum of six terms, commencing at the third.

12. The first term of a geometric progression is 5 and the ratio 2. How many terms of this series must be taken that their sum may be equal to 33 times the sum of half as many terms?

CHAPTER IX.

ARRANGEMENTS AND GROUPS.

1. DEFINITIONS. Every different order in which given things can be placed is called an *Arrangement* or *Permutation*, and every different selection that can be made is called a *Group* or *Combination*.

Thus if we take the letters a, b, c two at a time there are six arrangements, viz:

$$ab,\ ac,\ ba,\ bc,\ ca,\ cb,$$

but there are only three groups, viz:

$$ab,\ ac,\ bc.$$

If we take the letters a, b, c *all* at a time, there are six arrangements, viz:

$$abc,\ acb,\ bac,\ bca,\ cab,\ cba,$$

but there is only one group, viz:

$$abc.$$

2. PROBLEM. TO FIND THE NUMBER OF ARRANGEMENTS OF n DIFFERENT THINGS TAKEN ALL AT A TIME.

First. If we take one thing, say the letter a, there can be but one arrangement, viz: the thing itself.

Second. If we take two things, say the letters a and b, there are two arrangements, viz:

$$ab,\ ba.$$

Third. If we take three things, say a, b, c, there are six arrangements, viz:

$$abc,\ acb,\ bac,\ bca,\ cab,\ cba.$$

Notice that there are two arrangements in which a stands first, two more in which b stands first, and two more in which c stands first.

Fourth. If we take four things, say a, b, c, d, then we may arrange the *three* letters b, c, d in every possible way and place a before each arrangement, then arrange the three letters a, c, d in every possible way and place b before each arrangement, then arrange the three letters a, b, d in every possible way and place

the letter *c* before each arrangement, and finally arrange the three letters *a*, *b*, *c* in every possible way and place the letter *d* before each arrangement. It is evident that all four letters *a*, *b*, *c*, *d* appear in each arrangement thus formed, and it is also evident that the number of arrangements in which *a* stands first is exactly the same as the number in which *b* stands first, and so on.

Hence there are in all four times as many arrangements of four things taking all at a time as there are of three things taking all at a time, or there are four times six or twenty-four arrangements of four things taking all at a time.

In general, if we have *n* things, say the letters *a*, *b*, *c*, *d*, *e*, *f*, . . then we may suppose all the letters but *a* arranged in every possible order and then *a* placed before each of these arrangements; then we may suppose all the letters but *b* arranged in every possible order and then *b* placed before each of these arrangements, and so on.

It is evident that all *n* letters appear in each arrangement thus formed, and it is also evident that the number of arrangements in which *a* stands first is exactly the same as the number in which any other letter stands first.

Now the number of arrangements in which *a* stands first is evidently the number of arrangements of $(n-1)$ things taken all at a time, and hence the total number of arrangements of *n* things taking all at a time is *n* times the number of arrangements of $n-1$ things taking all at a time.

Let us represent the number of arrangements of *n* things taking all at a time by A_n and the number of arrangements of $n-1$ things taken all a time by A_{n-1}, etc. Then by what has just been shown we have

$$\begin{aligned}
A_n &= nA_{n-1}, \\
A_{n-1} &= (n-1)A_{n-2}, \\
A_{n-2} &= (n-2)A_{n-3}, \\
&\vdots \\
A_3 &= 3A_2, \\
A_2 &= 2A_1, \\
A_1 &= 1.
\end{aligned}$$

Now multiply these equations together, member by member, and we get

$$A_1\,A_2\,A_3\ .\ .\ .\ A_n=2A_1\ 3A_2\ .\ .\ .\ nA_{n-1}$$
$$=1\times2\times3\ .\ .\ .\ n\ A_1\ A_2\ .\ .\ .\ A_{n-1}.$$

By cancelling common factors we get

$$A_n=1\times2\times3\ .\ .\ .\ n.$$

The product of the integer numbers from n down to 1 or from 1 up to n is often represented by $\underline{|n}$ or $n!$, and is read factorial n, or n admiration.

With this notation we may write

$$A_n=\underline{|n}.$$

3. Problem. To find the Number of Arrangements of n Things taken r at a Time.

Let us first take a particular case, say the number of arrangements of five things, say the five letters a, b, c, d, e, taken three at a time. Suppose the arrangements all made and we select those which begin with a and put them by themselves in one class, then those which begin with b and put them by themselves in another class, and so on. We then divide the whole number of arrangements into five classes, and it is evident that the number in any one class is just the same as in any other class. Consider those which begin with a. Then every arrangement in this class contains besides a two of the four letters b, c, d, e, and since a is fixed and the other letters arranged in every possible way, therefore the number of *these* arrangements must equal the number of arrangements of the four letters b, c, d, e taken two at a time.

In general, if we have n things, say the letters $a, b, c, d, e, f,$ . . to be taken r at a time, we may select all those arrangements which begin with a and put them by themselves in one class, then those which begin with b and put them by themselves in another class, and so on. We thus divide the whole number of arrangements into n classes, and it is evident that the number of arrangements in any one class is just the same as the number of arrangements in any other class.

Consider those which begin with a.

Then every arrangement in this class contains besides a, $(r-1)$ of the letters b, c, d, . . , and since a is fixed while the remaining letters are arranged in every possible order, therefore the *number* of arrangements in the class considered must equal the number of arrangements of $n-1$ letters b, c, d, . . , taken $r-1$ at a time.

As there are n such classes and as the number of arrangements in each class equals the number of arrangements of $n-1$ things taking $r-1$ at a time, therefore the total number of arrangements of n things taken r at a time equals n times the number of arrangements of $n-1$ things taken $r-1$ at a time.

Let us represent the number of arrangements of n things taken r at a time by $A(^n_r)$ and similarily any number of things taken any number at a time, say s things taken t at a time (s being greater than t) by $A(^s_t)$, then by what has just been proved

$$A(^n_r)=nA(^{n-1}_{r-1})$$
$$A(^{n-1}_{r-1})=(n-1)A(^{n-2}_{r-2})$$
$$\vdots \qquad \vdots$$
$$A(^{n-r+1}_{1})=(n-r+1).$$

Multiply these equations together, member by member, and cancel common factors and we get

$$A(^n_r)=n(n-1)(n-2) \ . \ . \ . \ (n-r+1).$$

Multiply and then divide the right-hand member by $(n-r)(n-r+1)$ . . . 1 and we get

$$A(^n_r)=\frac{n(n-1)(n-2) \ . \ . \ . \ (n-r+1)(n-r)(n-r-1) \ . \ . \ . \ 1}{(n-r)(n-r-1 \ . \ . \ . \ 1}.$$

It is easily seen that the numerator is $\underline{|n}$ and the denominator is $\underline{|n-r}$, hence

$$\therefore A(^n_r)=\frac{\underline{|n}}{\underline{|n-r}}$$

4. Problem. To find the Number of Groups of n Different Things taken r at a Time.

Take the letters a, b, c, d, e, . . . , and suppose the groups all written down; then, fixing our attention upon any one group, it is evident that there could be several different arrangements made from that group by changing the order of the letters.

It is further evident that if we form all *possible* arrangements in *each* group we thereby obtain the total number of arrangements of the n letters taken r at a time.

The total number of arrangements then equals the number of arrangements in each group multiplied by the number of groups. Hence, representing the number of groups of n things taken r at a time by $G\binom{n}{r}$ and remembering that the number of arrangements in each group equals the number of arrangements of r things taken all at a time, that is $\underline{|r}$, and further remembering that the total number of arrangements equals

$$n(n-1)(n-2)\ .\ .\ (n-r+1),$$

we have

$$\underline{|r}\ G\binom{n}{r} = \frac{\underline{|n}}{\underline{|n-r}}$$

hence

$$G\binom{n}{r} = \frac{\underline{|n}}{\underline{|r}\ \underline{|n-r}}$$

5. The form of this result shows that the number of groups of u things taken r at a time is the same as the number taken $n-r$ at a time. This is also evident in another way, for every time we *select* r things from n things we *leave out* $n-r$ things; hence there must be as many ways of *leaving out* $n-r$ things as of *selecting* r things, but of course there are as many ways of *selectiug* $n-r$ things as there are of *leaving out* $n-r$ things.

6. In all that precedes, it was supposed that the given things were all different and that in forming the arrangements or groups none of the given things were repeated. Now we will consider arrangements and groups in which the things may be repeated and those in which the given things are not all alike.

7. PROBLEM. TO FIND THE NUMBER OF ARRANGEMENTS OF n THINGS TAKEN r AT A TIME, REPETITIONS BEING ALLOWED.

Suppose first we wish the number of arrangements, including repetitions, of the four letters a, b, c, d taken one at a time. Evidently there are four arrangements, viz: a, b, c, d.

Next suppose we wish the arrangements, including repetitions, of the four letters a, b, c, d taken two at a time.

The arrangements are the following:

$$\begin{array}{cccc} aa & ab & ac & ad \\ ba & bb & bc & bd \\ ca & cb & cc & cd \\ da & db & dc & dd \end{array}$$

Thus we see that there are sixteen arrangements, that is, 4^2 arrangements. In exactly the same way if we have n letters $a, b, c, d, e, f, \ldots$, the a may be followed by each of the n letters, giving n arrangements beginning with a; the b may be followed by each of the n letters, giving n arrangements beginning with b, etc. So there are evidently n arrangements beginning with *each* letter; hence in all there are n^2 arrangements of n things taken two at a time, allowing repetitions.

Let us now find the number of arrangements, allowing repetitions, of n things taken three at a time; and first to give definiteness to the ideas, consider the number of arrangements, allowing repetitions, of four letters a, b, c, d taken three at a time. We have written out the sixteen arrangements of four letters taken two at a time, and now we may suppose each of these sixteen arrangements to be preceded by the letter a, then each of these sixteen arrangements to be preceded by b, etc. We then have sixteen arrangements of three letters each, beginning with each letter, and as there are four letters there are in all four times sixteen, or sixty-four, arrangements of the letters a, b, c, d taken three at a time, repetitions being allowed.

Now, in the same way, if we have n letters $a, b, c, d, e, f, \ldots$, we may suppose each of the n^2 arrangements two at a time to be preceded by a, then each of these same n^2 arrangements to be preceded by b, etc.

Thus we get n^2 arrangements beginning with a, n^2 arrangements beginning with b, n^2 arrangements beginning with c, etc. Hence in all we obtain n times n^2, or n^3, arrangements of n letters taken three at a time, repetitions being allowed.

In general, if we know the number of arrangements of n letters taken s at a time, repetitions being allowed, we may find the number of arrangements of the n letters taken $s+1$ at a time.

Representing the number of arrangements, with repetitions, of

n letters a, b, c, d, e, f, . . . taken s at a time by N_s, we may then write a before each of these N_s arrangements s at a time and obtain N_s arrangements $s+1$ at a time beginning with a.

We may also write b before each of the same N_s arrangements and obtain N_s arrangements $s+1$ at a time beginning with b, and so on until each of the n letters a, b, c, d, . . is in turn placed before *each* of the N_s arrangements s at a time, and we then obtain nN_s arrangements taken $s+1$ at a time, repetitions being allowed.

Represent this number by N_{s+1} and we have

$$N_{s+1}=nN_s,$$

s being a positive integer which may be greater or less than n.

Giving s in turn all intermediate values from $r-1$ down to 1 and remembering that the number of arrangements one at a time is equal to n, we have

$$\begin{aligned} N_r&=nN_{r-1}\\ N_{r-1}&=nN_{r-2}\\ N_{r-2}&=nN_{r-3}\\ &\vdots\\ N_2&=nN_1\\ N_1&=n. \end{aligned}$$

Multiply these equals together and cancel the common factors and we get

$$N_r=n^r.$$

8. Problem. To find the Number of Groups of n Things Taken r at a Time, Repetitions being Allowed.

To prepare the way for the general case we begin with the groups of the four letters a, b, c, d taken three at a time, repetitions being allowed.

In this case there are twenty groups, vis:

aaa	*aab*	*aac*	*aad*	*abb*
abc	*abd*	*acc*	*acd*	*add*
bbb	*bbc*	*bbd*	*bcc*	*bcd*
bdd	*ccc*	*ccd*	*cdd*	*ddd*

Now if in *each* of these twenty groups we leave the first letter standing and advance the second letter one step and the third letter two steps, we get twenty new groups of the six letters a, b, c, d, e, f, as follows:

abc abd abe abf acd
ace acf ade adf aef
bcd bce bcf bde bdf
bef cde cdf cef def

The groups here written are the groups of the *six* letters a, b, c, d, e, f, *without repetitions.*

In a similar manner we may deal with the general case of the number of groups of n letters $a, b, c, d, e, f, \ldots$ taken r at a time, repetitions being allowed. Let the number of these groups be denoted by N_r and suppose them all written down in alphabetical order; then in *each* of these groups keep the first letter unchanged, advance the second letter one step, the third letter two steps, the fourth letter three steps and so on.

We thus form N_r new groups containing all the letters the original ones contained, and $r-1$ other letters. These new groups are written in alphabetical order, because the original ones were, and by the way in which the letters have been advanced it is evident that no letter is repeated in any one of these new groups.

No two of these new groups are alike, else two of the original groups would have been alike.

Now since each of these new groups contain r of the $n+r-1$ letters $a, b, c, d, e, \ldots$, and since no letter is repeated in any group, and since no two groups are alike, therefore these new groups constitute some or all of the groups of the $n+r-1$ letters $a, b, c, d, e, \ldots$ taken r at a time without repetitions.

Let the number of groups without repetitions of $n+r-1$ things taken r at a time be represented by $G\binom{n+r-1}{r}$ then it is evident that N_r cannot *exceed* $G\binom{n+r-1}{r}$. Now let us conceive each of the $G\binom{n+r-1}{r}$ groups written down in alphabetical order, and then leave the first letter in each group unchanged, change the second letter in each group to the one just before it in the alphabet, the third one in each group to the second one before it in the alphabet and so on, then these groups are changed into new groups wherein some of the letters are repeated, but no letter is beyond the nth letter of the alphabet. Moreover no two of these groups
Since are alike, ~~hence~~ no two of those from which they were formed were alike, so that these new groups must be some or all of the n letters $a, b, c, d, e, \ldots$ taken r at a time with repetitions.

These last formed groups are $G\binom{n+r-1}{r}$ in number, being formed from that number of groups, and as the number of groups with repetitions of n things taken r at a time has already been represented by N_r, hence $G\binom{n+n-1}{r}$ cannot exceed N_r.

It was previously proved that N_r could not exceed $G\binom{n+r-1}{r}$, hence, since neither can exceed the other, the number must be the same, or, in other words, the number of groups of n things taken r at a time, repetitions being allowed, is equal to the number of groups of $(n+r-1)$ things taken r at a time without repetitions. The last number has already been found. Hence the number of groups of n things taking r at a time, repetitions being allowed, equals

$$\frac{(n+r-1)(n+r-2)\ \ldots\ n}{\underline{|r}}$$

which may be written in either of the forms

$$\frac{n(n+1)\ \ldots\ (n+r-1)}{\underline{|r}}$$

or

$$\frac{\underline{|n+r-1}}{\underline{|n-1}\ \underline{|r}}$$

9. Problem. To find the Number of Arrangements where the Given Things are not all Different.

Illustration.—From what has gone before we know that the number of arrangements of the letters a, b, c, d taken all at a time is twenty-four, but if we have the letters a, a, b, c the number of arrangements is only twelve. These twelve are the following:

aabc aacb abac abca
acab acba baac baca
bcaa caab caba cbaa

If we have the letters a, a, b, b there are only six arrangements, viz:

aabb abab abba
baab baba bbaa

If we have the letteas a, a, a, b there are only four arrangements, viz:

aaab aaba abaa baaa.

Thus we see that with a given number of things the number of arrangements depends upon how many of each kind are alike.

Suppose now we have in all n letters, of which a is repeated r times, b is repeated s times, c is repeated t times, and so on so that $r+s+t+\ldots=n$, and we wish to find the number of arrangements taking all the n letters at a time.

Fixing our attention upon any arrangement whatever of the n letters, let all the letters but the a's remain unchanged while the r a's change places among themselves. Because all these a's are alike we get only one arrangement, but if they had all been different we would have obtained $\underline{|r}$ arrangements, and since the same thing is true whatever the arrangement upon which we fixed our attention to begin with, it follows that there are $\underline{|r}$ times as many arrangements when all the r letters are different as there are under the present supposition. In the same way there are $\underline{|s}$ times as many arrangements when the s b's are all different as there are under the present supposition, also there are $\underline{|t}$ times as many arrangements when the t c's are all different as there are under the present supposition, and so on.

Hence there are $\underline{|r}\,\underline{|s}\,\underline{|t}\ldots$ times as many arrangements when the n letters all are different as there are under the present supposition, or the number of arrangements under the present supposition is equal to the number of arrangements of n things taken all at a time, when all are different, divided by $\underline{|r}\,\underline{|s}\,\underline{|t}\ldots$, that is, the number of arrangements under the present supposition is equal to

$$\frac{\underline{|n}}{\underline{|r}\,\underline{|s}\,\underline{|t}\ldots}$$

10. PROBLEM. TO FIND THE NUMBER OF WAYS IN WHICH n THINGS, NO TWO ALIKE, CAN BE MADE UP INTO SETS of which the first set contains r things, the second set contains s things, the third contains t things, and so on, where of course

$$r+s+t+\ldots=n.$$

We begin with a special case and find the number of ways five letters a, b, c, d, e, can be made up into two sets of which the first set contains two, and the second set three letters.

Consider any particlar way of dividing into sets, say the first set is ab, and the second set is cde. Then keeping the *sets* undis-

turbed, there could be twelve arrangements made from this one divisions into sets. The twelve arrangements are:

ab cde	*ba cde*
ab ced	*ba ced*
ab dce	*ba dce*
ab dec	*ba dec*
ab ecd	*ba ecd*
ab edc	*ba edc*

From any *other* way of dividing into sets there could be twelve arrangements found, hence the whole number of arrangements of five letters equals twelve times the number of ways of dividing into sets, or the number of sets equals one-twelfth the number of arrangements. The number of arrangements in this case is $\underline{|5}$, hence the number of ways of makimg up sets in this case equals $\frac{1}{12}\underline{|5} = 10$.

We will now take the general case of n letters $a, b, c, d, e, f, \ldots$ and take the first r letters to form the first set, the following s letters to form the second set, the next following t letters for the third set and so on.

Place the letters of the first set down in a horizontal line, then those of the second set in the *same* horizontal line *following* the first set, and those of the third set in the same horizontal line following those of the second set, and so on.

We thus have all the n letters arranged in a horizontal line, and it is evident that we could keep these *sets* undisturbed, but still make several arrangements of the n letters in a horizontal line. The letters in the first set can be arranged in $\underline{|r}$ ways, those of the second set in $\underline{|s}$ ways, those of the third set in $\underline{|t}$ ways, and so on, and as any arrangement in *any* set may accompany *any* arrangement in *any other* set, hence the whole number of arrangements while the sets are undisturbed is equal to $\underline{|r}\,\underline{|s}\,\underline{|t} \ldots$

Thus from *one* way of making up the sets there are $\underline{|r}\,\underline{|s}\,\underline{|t} \ldots$ arrangements, and of course from any other division into sets, there could be formed the same number of arrangements, hence the whole number of arrangements of n things, all at a time, equals $\underline{|r}\,\underline{|s}\,\underline{|t} \ldots$ times the number of ways of making up the sets, or the number of ways of making up the sets, equals the

number of arrangements divided by $\underline{|r}\,\underline{|s}\,\underline{|t}\ldots$, or the number of ways of making up n things into sets, of which the first contains r things, the second s things, the third t things, and so on, equals

$$\frac{\underline{|n}}{\underline{|r}\,\underline{|s}\,\underline{|t}\ldots}$$

11. Given a set of K things, another set of L things, another of M things, and so on; to find the Number of Groups that can be Made by taking r Things from the First set, s Things from the Second set, t from the Third set, and so on.

Of the K things taken r at a time there are $\dfrac{\underline{|K}}{\underline{|r}\,\underline{|K-r}}$ groups,

and of the L things taken s at a time there are $\dfrac{\underline{|L}}{\underline{|s}\,\underline{|L-s}}$ groups,

and of the M things taken t at a time there are $\dfrac{\underline{|M}}{\underline{|t}\,\underline{|M-t}}$ groups,

and so on, and as any one of the groups from the first set may be taken with any one of the groups from the second set, and any one from the third set, and so on, to form a larger group, it follows that the total number of these larger groups equals the product

$$\frac{\underline{|K}}{\underline{|r}\,\underline{|K-r}}\;\frac{\underline{|L}}{\underline{|s}\,\underline{|L-s}}\;\frac{\underline{|M}}{\underline{|t}\,\underline{|M-t}}\ldots$$

12. There are various relations connecting arrangements with arrangements, groups with groups, arrangements with groups, etc. We will obtain a few of these relations, and recommend that the student try to obtain others not here given.

One relation was obtained in Art. 2, where it was shown that

$$\mathrm{A}\binom{n}{n}=n\mathrm{A}\binom{n-1}{n-1}, \qquad (1)$$

and another in Art. 3, where it was shown that

$$\mathrm{A}\binom{n}{r}=n\mathrm{A}\binom{n-1}{r-1}. \qquad (2)$$

We have already found that

$$\mathrm{A}\binom{n}{r}=\frac{\underline{|n}}{\underline{|n-r}} \qquad (3)$$

and from this it follows that

$$A(\tbinom{n}{r-1})=\frac{\lfloor n}{\lfloor n-r+1} \qquad (4)$$

But $\lfloor n-r+1$ equals the product of the integer numbers from 1 up to $n-r+1$, and this product of course equals $n-r+1$ times the product of the integer numbers from 1 up to $n-r$, or

$$\lfloor n-r+1=(n-r+1)\lfloor n-r,$$

hence
$$A(\tbinom{n}{r-1})=\frac{\lfloor n}{(n-r+1)\lfloor n-r} \qquad (5)$$

Comparing this with the value of $A(\tbinom{n}{r})$, equation (3), we get

$$A(\tbinom{n}{r})=(n-r+1)A(\tbinom{n}{r-1}). \qquad (6)$$

If in (3) we make $r=n$ we get

$$A(\tbinom{n}{n})=A(\tbinom{n}{n-1}), \qquad (7)$$

or the number of arrangements of n things taken all at a time equals the number of arrangements of n things all but one at a time.

We have already found that

$$G(\tbinom{n}{r})=\frac{\lfloor n}{\lfloor r\,\lfloor n-r} \qquad (8)$$

and from this it follows that

$$G(\tbinom{n-1}{r})=\frac{\lfloor n-1}{\lfloor r\,\lfloor n-r-1} \qquad (9)$$

Multiply both numerator and denominator of this last fraction by $n(n-r)$, remembering that $n\lfloor n-1=\lfloor n$ and that $(n-r)\lfloor n-r-1=\lfloor n-r$, and we get

$$G(\tbinom{n-1}{r})=\frac{(n-r)\lfloor n}{n\lfloor r\,\lfloor n-r} \qquad (10)$$

hence, from (8) and (9),

$$G(\tbinom{n}{r})=\frac{n}{n-r}G(\tbinom{n-1}{r}). \qquad (11)$$

From (8) it easily follows that

$$G(\tbinom{n}{r-1})=\frac{\lfloor n}{\lfloor r-1\,\lfloor n-r+1} \qquad (12)$$

Multiply both numerator and denominator of this last fraction by r and remember that $r\underline{|r-1}=\underline{|r}$ and that $\underline{|n-r+1}=(n-r+1)\underline{|n-r}$, we get

$$G\binom{n}{r-1}=\frac{r\underline{|n}}{(n-r+1)\underline{|r}\,\underline{|n-r}} \qquad (13)$$

Comparing (13) and (8) we easily get

$$G\binom{n}{r}=\frac{n-r+1}{r}G\binom{n}{r-1}. \qquad (14)$$

From (8) it easily follows that

$$G\binom{n-1}{r-1}=\frac{\underline{|n-1}}{\underline{|r-1}\,\underline{|n-r}} \qquad (15)$$

From (15) and (9) we get

$$G\binom{n-1}{r}+G\binom{n-1}{r-1}=\frac{\underline{|n-1}}{\underline{|r}\,\underline{|n-r+1}}+\frac{\underline{|n-1}}{\underline{|r-1}\,\underline{|n-r}}$$

$$\frac{(n-r)\,\underline{|n-1}}{\underline{|r}\,\underline{|n-r}}+\frac{r\,\underline{|n-1}}{\underline{|r}\,\underline{|n-r}}=\frac{n\,\underline{|n-1}}{\underline{|r}\,\underline{|n-r}}=\frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}},$$

which by Art. (8) equals $G\binom{n}{r}$, hence

$$G\binom{n}{r}=G\binom{n-1}{r}+G\binom{n-1}{r-1}. \qquad (16)$$

We have obtained a few relations connecting arrangements with arrangements in equations (1), (2), (6), (7), also a few relations connecting groups with groups in equations (11), (14), (16). We now obtain a few relations involving both arrangements and groups in the same equation.

We have already found in Art. 4

$$A\binom{n}{r}=\underline{|r}\,G\binom{n}{r}, \qquad (17)$$

and as $\underline{|r}=A\binom{r}{r}$ we may write (17) in the form

$$A\binom{n}{r}=A\binom{r}{r}G\binom{n}{r}. \qquad (18)$$

From (7), $A\binom{r}{r}=A\binom{r}{r-1}$ and writing this value in (18) we get

$$A\binom{n}{r}=A\binom{r}{r-1}G\binom{n}{r}. \qquad (19)$$

In (18) substitute the value of $G\binom{n}{r}$ given in (16) and we get

$$A\binom{n}{r}=A\binom{r}{r}\left[G\binom{n-1}{r}+G\binom{n-1}{r-1}\right]. \qquad (20)$$

But it readily follows from (18) that

$$A\binom{n-1}{r}=A\binom{r}{r}G\binom{n-1}{r},$$

or

$$G\binom{n-1}{r}=\frac{A\binom{n-1}{r}}{A\binom{r}{r}}.$$

Substitute this value of $G\binom{n-1}{r}$ in (20) and we get

$$A\binom{n}{r}=A\binom{n-1}{r}+A\binom{r}{r}G\binom{n-1}{r-1}. \qquad (21)$$

Since by Art. 8, groups where repetitions are allowed can be expressed in terms of groups when repetitions are not allowed, it would be an easy matter to obtain equations involving groups with repetitions.

13. Examples and Problems.

1. How many different groups of two each can be made from the letters a, d, l, n, s? See VIII, Art. 5.

2. How many arrangements of five each can be made from the letters of the word *Groups*?

3. How many different signals can be made with five flags of different colors hoisted one above another all at a time?

4. How many different signals can be made from seven flags of different colors hoisted one above another, five at a time?

5. How many different groups of 13 each can be made out of 52 cards, no two alike?

6. How many different signals can be made from five flags of different colors, which can be hoisted any number at a time above one another?

7. How many different signals can be made from seven flags of which 2 are red, 1 white, 3 blue, 1 yellow when all are displayed together, one above another, for each signal.

8. A certain lock opens for some arrangement of the numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, taken 6 at a time, repetitions allowed. How many trials must be made before we would be *sure* of opening the lock?

9. In how many ways can a committee of 3 appointed from 5 Germans, 3 Frenchmen and 7 Americans, so that each nationality is represented?

10. How many different arrangements can be made of nine ball players, supposing only two of them can catch and one pitch?

11. How many different products of three each can be made from the four letters a, b, c, d?

12. In how many different ways can the letters of the word *algebra* be written, using all the letters?

13. In how many ways can a child be named, supposing that there are 400 different Christian names, without giving it more than three Christian names?

14. In how many ways can seven people sit at a *round* table?

15. There are 5 straight lines in a plane, no two parallel; how many intersections are there?

16. On a railway there are 20 stations of a certain class. Find the number of different kinds of tickets required, in order that tickets may be sold at each station for each of the others.

17. Find the number of signals that can be made with four lights of different colors, which can be displayed any number at a time, arranged either above one another, side by side, or diagonally.

18. From a company of 90 men, 20 are detached for mounting guard each day; how long will it be before the same 20 men are on guard together, supposing the men to be changed as much as possible? How often will each man have been on guard during this time?

19. A lock contains 5 levers, each capable of being placed in 10 distinct positions. At a certain arrangement of the levers the lock is open. How many locks of this kind can be made so that no two shall have the same key?

20. There are n points in a plane no three of which are in the same straight line. Find the number of straight lines which result from joining them.

21. There are n points in a plane, no three of which are in the same straight line except r, which are all in the same straight line; find the number of straight lines which result from joining them.

22. There are n points in space, no four of which are in the same plane with the exception of r which are all in the same plane. How many planes are there, each containing three of the points?

CHAPTER X.

BINOMIAL THEOREM.

1. The Binomial Theorem enables us to find any power of a binomial without the labor of obtaining the previous powers. In order to observe the law of formation of a power of a binomial we first observe the law of formation of the product of several binomial factors of the form $x+a$, $x+b$, $x+c$, etc., and we will afterwards be able to arrive at the power of a binomial by the supposition that $a=b=c$, etc.

2. LAW OF THE PRODUCT OF FACTORS OF THE FORM $x+a$, $x+b$, $x+c$, ETC.

By actual multiplication it is seen that

$$(x+a)(x+b)=x^2+(a+b)x+ab,$$
$$(x+a)(x+b)(x+c)=x^3+(a+b+c)x^2+(ab+ac+bc)x+abc,$$
$$(x+a)(x+b)(x+c)(x+d)=x^4+(a+b+c+d)x^3+$$
$$(ab+ac+ad+bc+bd+cd)x^2+(abc+abd+acd+bcd)x+abcd.$$

By a careful inspection of these products we will discover the presence of two uniform laws—a law for the exponents and a law for the coefficients.

The law of the exponents is readily seen to be as follows:

The exponent of x in the first term of the product is equal to the number of binomial factors, and in the remining terms it continually decreases by one until it is zero.

The law of the coefficients may be stated thus:

The coefficient of the first term is unity; the coefficient of the second term is the sum of the second terms of the binomial factors; the coefficient of the third term is the sum of all their different products taken two at a time; the coefficient of the fourth term is the sum of all their different products taken three at a time, and so on. The last term is the product of all the second terms of the binomial factors.

3. PROOF THAT THE LAWS ARE GENERAL. We will now show that if the laws observed above hold in the product of a given number of binomial factors, they will hold in the product of any number of binomial factors whatever.

For, assume that we have tested the above laws in the case of the product of a certain number of factors, suppose n, and have found them to hold true.

To facilitate the discussion we will represent the n second terms of the binomial factors by $a_1, a_2, a_3, a_4, \ldots a_n$* instead of a, b, c, d, etc., and accordingly the product of the n binomials

$$(x+a_1)(x+a_2)(x+a_3)(x+a_4)\ldots(x+a_{n-1})(x+a_n)$$
$$=x^n+(a_1+a_2+a_3+a_4\ldots+a_n)x^{n-1}$$
$$+(a_1a_2+a_1a_3+a_1a_4+\ldots+a_{n-1}a_n)x^{n-2}$$
$$+(a_1a_2a_3+a_1a_2a_4+a_1a_2a_5+\ldots+a_{n-2}a_{n-1}a_n)x^{n-3}$$
$$+\ldots+a_1a_2a_3a_4\ldots a_n.$$

In order to abreviate this expression it is convenient to let

P_1 equal the *first parenthesis*, or the sum of all the different second terms of the binomial factors.

P_2 equal the *second parenthesis*, or the sum of all the different products of the second terms of the binomial factors taken *two* at a time.

P_3 equal the *third parenthesis*, or the sum of all the different products of the second terms of the binomial factors taken *three* at a time; and so on.

P_n equal the *n th parenthesis*, or the product of all the second terms of the binomial factors.

With these abbreviations the second member of the above equation reads

$$x^n+P_1x^{n-1}+P_2x^{n-2}+P_3x^{n-3}+\ldots+P_n.$$

Multiplying this expression, which represents the product of n binomial factors, by a new binomial, $x+a_{n+1}$, we derive the following result for the product of $n+1$ binomial factors:

$$x^{n+1}+(P_1+a_{n+1})x^n+(P_2+a_{n+1}P_1)x^{n-1}$$
$$+(P_3+a_{n+1}P_2)x^{n-2}+\ldots+a_{n+1}P_n.$$

It is seen from this result that the law of exponents still holds. For there are $n+1$ binomials and the exponent of x begins, in the first term, with $n+1$ and decreases continually by one through the remaining terms until the value zero is reached.

*This notation presents many mechanical advantages. It must not be supposed, however, that there is any relation subsisting between a_1 and a_2 or any other two of the symbols; they are as independent as distinct letters.

The law of coefficients holds good also. For:

The coefficient of the first term is unity.

The coefficient of the second term is P_1+a_{n+1}. Now P_1 stands for the sum of the n terms $a_1+a_2+a_3+ \ldots a_n$. Hence P_1+a_{n+1}, or the coefficient of the second term, is the sum of all the different second terms of the binomial factors.

The coefficient of the third term is $P_2+a_{n+1}P_1$.

Now P_2 represents the sum of all the different products of the n letters $a_1, a_2, a_3, a_4, \ldots a_n$ taken two at a time.

(1). That is, P_2 represents the sum of all the different products of the $n+1$ letters $a_1, a_2, a_3, \ldots a_{n+1}$ taken two at a time *which do not contain* a_{n+1}.

Again,

$$a_{n+1}P_1=a_1a_{n+1}+a_2a_{n+1}+a_3a_{n+1}+ \ldots +a_na_{n+1}.$$

(2). That is, $a_{n+1}P_2$ equals the sum of all the different products of the $n+1$ letters $a_1, a_2, a_3, a_4, \ldots a_{n+1}$, taken two at a time, *which contain* a_{n+1}.

Therefore, putting *(1)* and *(2)* together, $P_2+a_{n+1}P_1$ equals the sum of all the different products of the $n+1$ letters $a_1, a_2, a_3, a_4, \ldots a_{n+1}$, taken two at a time, both those which do and those which do not contain a_{n+1}.

The coefficient of the fourth term is $P_3+a_{n+1}P_2$.

Now P_3 equals the sum of all the different products of the n letters $a_1, a_2, a_3, a_4, \ldots a_n$, taken three at a time.

(3). That is, P_3 equals the sum of all the different products of the $n+1$ letters $a_1, a_2, a_3, a_4, \ldots a_{n+1}$, taken three at a time, *which do not contain* a_{n+1}.

Again,

$$a_{n+1}P_2=a_1a_2a_{n+1}+a_1a_3a_{n+1}+a_1a_4a_{n+1}+ \ldots +a_{n-1}a_na_{n+1}.$$

(4). That is, $a_{n+1}P_2$ equals the sum of all the different products of the $n+1$ letters $a_1, a_2, a_3, a_4, \ldots a_{n+1}$, taken three at a time, *which contain* a_{n+1}.

Therefore, putting *(3)* and *(4)* together, $P_3+a_{n+1}P_2$ equals the sum of all the different products of the $n+1$ letters $a_1, a_2, a_3, a_4, \ldots a_{n+1}$, taken three at a time.

In like manner we may treat the coefficient of the fifth term, and so on. The last term is the product of all the $n+1$ letters $a_1, a_2, a_3, a_4, \ldots a_{n+1}$.

Therefore, we have proved that if the laws of exponents and coefficients hold in the product of n factors, they will hold also in the product of $n+1$ factors.

But they have been proved by actual multiplication to hold when four factors are multiplied together, therefore they hold when five factors are multiplied together, and if they hold when five factors are multiplied together they must hold when six are multiplied together, ana so on indefinitely. Hence the laws hold universally.

4. DEDUCTION OF THE BINOMIAL FORMULA.

We have now proved that the equation

$$\begin{aligned}(x+a_1)(x+a_2)(x+a_3)\ .\ .\ .\ (x+a_{n-1})(x+a_n)\\ =x^n+(a_1+a_2+a_3+\ .\ .\ .\ +a_n)x^{n-1}\\ +(a_1a_2+a_1a_3+a_1a_4+\ .\ .\ .\ +a_{n-1}a_n)x^{n-2}\\ +(a_1a_2a_3+a_1a_2a_4+a_1a_2a_5+\ .\ .\ +a_{n-2}a_{n-1}a_n)x^{n-3}\\ +\ .\ .\ .\ +a_1a_2a_3a_4\ .\ .\ a_n\end{aligned}$$

is true for all positive values of n.

Since a_1, a_2, a_3, a_4, . . . a_n are any numbers whatever, we may assume that they are all alike and we may suppose each equal to the quantity a. Then each of the factors in the left-hand side of the above equation will become equal to $x+a$, and consequently the left-hand member will become

$$(x+a)^n.$$

On the right-hand side of the equation the term x^n remains unchanged. In the second term the parenthesis becomes the sum of n a's; that is, it is equal to na, so that the second term itself becomes

$$nax^{n-1}.$$

In the third term the parenthesis reduces to the sum of as many a^2's as there are groups of n things taken two at a time: that is, the parenthesis becomes $\frac{n(n-1)}{1\,.\,2}a^2$, so that the third term itself becomes

$$\frac{n(n-1)}{1\,.\,2}a^2x^{n-2}.$$

In the fourth term the parenthesis reduces to the sum of as many a^3's as there are groups of n things taken three at a time;

that is, the parenthesis becomes $\frac{n(n-1)(n-2)}{1.2.3}a^3$, so that the fourth term itself becomes

$$\frac{n(n-1)(n-2)}{1.2.3}a^3x^{n-3},$$

and so on for the other terms.

The last term reduces to the product of n a's; that is, to

$$a^n.$$

Therefore, on the supposition that $a_1=a_2=a_3= \ldots =a_n$, the equation above written becomes

$$(x-a)^n=$$

$$x^n+nax^{n-1}+\frac{n(n-1)}{1.2}a^2x^{n-2}+\frac{n(n-1)(n-2)}{1.2.3}a^3x^{n-3}+\ldots+a^n,$$

which is the *Binomial Formula.*

The expression on the right-hand side of the equation is called the *Expansion* or the *Development* of the power of the binomial.

5. EXAMPLE. Expand $(y+2)^5$.

Substitute y for x, 2 for a, and 5 for n, in the binomial formula and we obtain

$$(y+2)^5=y^5+5.2y^4+\frac{5.4}{1.2}(2)^2y^3+\frac{5.4.3}{1.2.3}(2)^3y^2+\frac{5.4.3.2}{1.2.3.4}(2)^4y+(2)^5,$$

or simplifying,

$$(y+2)^5=y^5+10y^4+40y^3+80y^2+80y+32.$$

6. BINOMIAL THEOREM. The binomial formula may be stated in the form of a theorem as follows:

In any power of a binomial $x+a$, the exponent of x begins in the first term with the exponent of the power, and in the following terms continually decreases by one. The exponent of a commences with one in the second term of the power, and continually increases by one.

The coefficient of the first term is one, that of the second is the exponent of the power; and if the coefficient of any term be multiplied by the exponent of x in that term and divided by the exponent of a increased by one, it will give the coefficient of the succeeding term.

7. HISTORICAL NOTE. The first rule for obtaining the powers of a binomial seems to have been given by Vieta (1540–1603). He observed as a necessary result of the process of multiplication that the successive coefficients of *any* power of a binomial are: first, unity; second, the sum of the first and

second coefficients in the preceding power, third, the sum of the second and third coefficients in the preceding power, and so on. Vieta noticed also the uniformity in the product of binomial factors of the form $x+a$, $x+b$, $x+c$, etc. But Harriot (1560-1621) independently and more fully treated of these products in showing the nature of the composition of a rational integral equation. See VI, Art. 1. In this connection it is interesting to note that Harriot was the first mathematician to transpose all the terms of an equation to the left member.

The binomial formula as now used; that is, the expansion of the nth power of a binomial, expressed with *factorial coefficients*, was the discovery of Sir Isaac Newton (1642-1727) and for that reason it is commonly called *Sir Isaac Newton's Binomial Theorem.*

8. NUMBER OF TERMS IN THE EXPANSION. The exponents of a through the binomial formula constitute the following scale:

$$0,\ 1,\ 2,\ 3,\ 4,\ \ldots\ n.$$

The number of terms in this scale is $n+1$. Therefore the number of terms in the expansion of $(x+a)^n$ is $n+1$.

9. VALUE OF THE rTH TERM. The value of the rth term in the expansion of $(x+a)^n$ can be easily found

By the law of exponents, the exponent of x in the *first* term is n; in the *second*, $n-1$; in the *third*, $n-2$, and so on; consequently in the rth term it is $n-(r-1)$, or $n-r+1$. Also by the law of the exponents, the exponent of a in the second term is 1; in the third term, 2, and so on; consequently in the rth term it is $r-1$. So, without the coefficient, the rth term must be $a^{r-1}x^{n-r+1}$.

By inspection of the coefficients in the expansion in Art. 4, it is seen that the numerator of the coefficient of any term is the product of the natural numbers from n to a number one greater than the exponent of x. Since the exponent of x in the rth term has been found to be $n-r+1$, this numerator of the coefficient must be $n(n-1)(n-2)\ \ldots\ (n-r+2)$.

An inspection of the binomial formula will also show that the denominator of any coefficient is the product of the natural numbers from unity to a number equal to the exponent of a. Whence the denominator of the coefficient of the rth term must be $1, 2, 3, \ldots (r-1)$. Therefore the complete rth term is

$$\frac{n(n-1)(n-2)\ \ldots\ (n-r+2)}{1.2.3.4\ \ldots\ (r-1)}a^{r-1}x^{n-r+1}.$$

Multiplying numerator and denominator of the coefficient by $\lfloor n-r+1$, this becomes

$$\frac{\lfloor n}{\lfloor r-1 \;\; \lfloor n-r+1} a^{r-1}x^{n-r+1}.$$

10. THEOREM. *In the expansion of $(x+a)^n$ the coefficient of the rth term from the beginning equals the coefficient of the rth term from the end.*

Since there are $n+1$ terms all together (Art. 8), the tth term from the end has $n+1-t$, or $n-t+1$, terms before it. Hence the tth term from the end is the same as the $n-t+2$ th term from the beginning. From the preceding article the $n-t+2$ th term equals

$$\frac{\lfloor n}{\lfloor n-t+1 \;\; \lfloor t-1} x^{n-t+1}a^{t-1}.$$

But from the preceding article the tth term from the beginning equals

$$\frac{\lfloor n}{\lfloor t-1 \;\; \lfloor n-t+1} a^{t-1}x^{n-t+1}.$$

It is plainly seen that the coefficients are identical.

11. EXPANSION OF $(x-a.)^n$

If we substitute $-a$ for a in the binomial formula we will obtain the following result for the expansion of $x-a$:

$$(x-a)^n = x^n - nax^{n-1} + \frac{n(n-1)}{1.2}a^2x^{n-2} - \frac{n(n-1)(n-2)}{1.2.3}a^3x^{n-3} + \,.\,.$$

12. THEOREM. *In the binomial formula the snm of the coefficients of the even terms equals the sum of the coefficients of the oda terms.*

In the expansion of $(x-a)^n$ put $x=1$ and $a=1$. We then obtain

$$0 = 1 - n + \frac{n(n-1)}{1.2} - \frac{n(n-1)(n-2)}{1.2.3} + \text{etc.},$$

which shows, since the negative on the right side of this equation must equal the positive, that the sum of the coefficients of the first, third, fifth, . . . terms equals the sum of the coefficients of the second, fourth, sixth, . . . terms.

13. THEOREM. *The sum of all the coefficients in the expansion of $(x+a)^n$ equals 2^n.*

In the expansion of $(x+a)^n$ put $x=1$ and $a=1$. We then have

$$2^n=1+n+\frac{n(n-1)}{2}+\frac{n(n-1)(n-2)}{1.2.3}+\text{etc},$$

14. EXAMPLES.

1. Expand $(x+a)^6$.

2. Expand $(b-c)^5$.

3. Expand $(y+3)^7$.

4. Expand $(b^2-c^2)^5$.

5. Expand $(x+a)^8$.

6. Expand $(x+2c)^5$.

7. Expand $(3b+\frac{1}{2})^6$.

8. Expand $(x^2+a^2)^7$.

9. Expand $(2ax-x^2)^4$.

10. Expand $(\sqrt{ab}-\sqrt[3]{ab})^6$.

11. Expand $\left[y^2+\frac{c^2}{y}\right]^6$.

12. Expand $(5-\frac{1}{8}x)^{10}$.

13. Find the 5th term of $(xy+x^2)^{12}$.

14. Find the 9th term of $\left[\frac{3}{2}x^{\frac{2}{8}}+\frac{2}{8}x^{\frac{3}{2}}\right]^{18}$.

15. Find the nth term of $\left[n^n+\frac{1}{n^n}\right]^n$.

16. Expand $(x^2+2ax+a^2)^3$.

17. Expand $(\sqrt{c^2-2x})^{12}$.

18. Find the 1000 term in $(x+a)^{2000}$.

15. EXPANSION OF A POLYNOMIAL. The power of a polynomial can be obtained in the following manner. Suppose it is required to expand $(a+b+c)^3$. We can proceed thus:

$$(a+b+c)^3=[a+(b+c)]^3$$
$$=a^3+3a^2(b+c)+3a(b+c)^2+(b+c)^3,$$

which, when the powers of $b+c$ are developed, becomes

$$a^3+3a^2b+3a^2c+3ab^2+6abc+3ac^2+b^3+3b^2c+3bc^2+c^3.$$

Notice that the result is a *homogeneous symmetrical function* of a, b, and c.

16. EXAMPLES.

1. Expand $(a+b-c)^2$.
2. Expand $(ab+bc+ac)^2$.
3. Expand $\left\{\dfrac{x^3-a^3}{x-a}\right\}^3$.
4. Expand $(a+b+c+d)^2$.
5. Expand $(1+x+x^2)^4$.

CHAPTER XI.

THEORY OF LIMITS.

1. Definition. When a quantity preserves its value unchanged in the same discussion it is called a *Constant*, but when under the conditions of the problem a quantity may assume an indefinite number of values it is called a *Variable*.

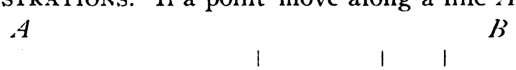
Constants are usually represented by the first or intermediate letters of the alphabet and variables by the last letters.

The notation by which we distinguish between constants and variables is the same as that by which we distinguish between known and unknown quantities, but it must not be thought that any analogy is intended to be pointed out by this fact. When we are discussing a problem in which both constants and variables appear we usually do not care whether the constants are known or unknown.

2. Definition. When a variable in passing from one value to another passes through all intermediate values it is called a *continuous* variable; when it does not pass through all intermediate values it is called a *discontinuous* variable.

3. Definition. When a variable so changes in value as to approach nearer and nearer some constant quantity which it can never equal, yet from which it may be made to differ by an amount as small as we please, this constant is called the *Limit* of the *variable*.

4. Illustrations. If a point move along a line AB, starting



at A and moving in such a way that the first second the point moves one-half the distance from A to B, the second second one-half the remaining distance, the third second one-half the distance which still remains, and so on; then the distance from A to the moving point is a variable whose limit is the distance AB. For, no matter how long the point has been moving, there is still *some*

distance remaining between it and the point B, so that the distance from A to the moving point can never equal AB, but as the moving point can be brought as near as we please to B, its distance from A can be made to differ from the distance AB by an amount as small as we please.

Thus we see that the distance from A to the moving point fulfills all the requirements of the definition of a variable, and the distance AB all the requirements of the definition of a limit.

The student must note that it is not the *point* B that is the limit of the *moving point*, although the moving point approaches the point B, but it is the *distance* AB. that is the limit of the *distance* from A to the moving point.

If we call the distance the point moves the first second 1 (then of course the whole distance AB would be 2), the distance traversed the second second would be $\frac{1}{2}$, that traversed the third second would be $\frac{1}{4}$, and so on, and the entire distance from A to the moving point at the end of n seconds would be the sum of n terms of the series

$$1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}, \ldots$$

Now it is sure that the more terms of this series that are taken the less does the sum differ from 2; but the sum can never equal 2. Hence we say that the *limit* of the sum of the series

$$1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16} \ldots$$

as the number of terms is indefinitely increased is 2.

Again consider any regular polygon inscribed in a circle, and then join the vertices with the middle points of the arcs subtending the sides, thus forming another regular inscribed polygon of double the number of sides. From *this* polygon form *another* of double *its* number of sides and so on. Now the polygon is always *within* the circle, and hence the *area* of the polygon can never equal the area of the circle, but as the process of doubling the number of sides is continued, the less does the area of the polygon differ from the area of the circle. Hence, we say that the limit of the area of the polygon is the area of the circle.

Again as a straight line is the shortest distance between two points, any side of the inscribed polygon is less than the subtended arc, hence the sum of all the sides or the perimeter of the polygon is less than the sum of all the subtended arcs or the cir-

cumference of the circle, or in other words the perimeter of the polygon can never equal the circumference of the circle, but as the process of doubling the number of sides is continued, the perimeter of the polygon differs less and less from the circumference of the circle, hence the circumference of the circle is the limit of the perimeter of the inscribed polygon.

5. The student should not infer from what has been said that *all* variables have limits. In fact, the truth is quite the contrary, for *most* variables do *not* have limits. Thus, in the illustration of the moving point given above, the variable does not have a limit if we suppose the point to move at a uniform rate. For, if the velocity is uniform, it is a mere question of time until the moving point *passes* B, or, in fact, any other point to the right of B, however remote. Much more would this be true if the point moved with increasing instead of uniform velocity.

Again, consider the fraction

$$\frac{x+1}{x}$$

If x be supposed to change in value, the value of the fraction changes and is itself a variable. Now suppose x to decrease in value. It is plain that *the value of the fraction increases without limit as x decreases.* In other words, the value of the fraction can be made as large as we please by taking x small enough. Hence, as x decreases, *the value of the fraction has no limit.*

6. It follows immediately from the definition of a variable, that the *difference between a variable and its limit is a variable whose limit is zero.*

For if x be a variable whose limit is a, then x may be made to differ from a by as small a quantity as we please, hence $a-x$ may be made as small as we please; yet as x can never equal a, $a-x$ can never equal zero, hence $a-x$ is a variable, whose limit is zero.

7. THEOREM. *If two variables are always equal and each approaches a limit, the limits must be equal.*

Let x and y be the variables, and let limit $x=a$ and limit $y=b$.

We are to prove that $a=b$. If a and b are not equal, suppose a greater than b and let

$$a-b=d$$

Let $$a-x=u \text{ and } b-y=v,$$
then $$a=x+u \text{ and } b=y+v,$$
and $a-b=d$ becomes by substitution,

$$(x+u)-(y+v)=d$$

or $$(x-y)+(u-v)=d$$

Since $\lim x=a$, $\lim u=o$ and as $\lim y=b$, $\lim v=o$, or u and v are each variables which can be made as small as we please, and hence the difference $u-v$ can be made as small as we please, and so can be made so small as not to cancel d; hence $x-y$ would equal something, or x and y would differ, which is contrary to the hypothesis; hence a cannot be greater than b, and in the same way it may be shown that b cannot be greater than a.

Therefore $$a=b.$$

8. THEOREM. *The limit of the algebraic sum of several variables equals the algebraic sum of their separate limits.*

Let the variables be x, y, z, etc., and let $\lim x=a$, $\lim y=b$, $\lim z=c$, etc., we are to prove

$$\lim (x+y+z+ \ldots)=(a+b+c+ \ldots)$$

Let
$$a-x=u \therefore x=a-u,$$
$$b-y=v \therefore y=b-v,$$
$$c-z=w \therefore z=c-w,$$
etc., etc.;

then $x+y+z+ \ldots =(a+b+c+ \ldots)-(u+v+w+ \ldots)$.

Suppose u to be numerically the greatest of the quantities u, v, w, ... and suppose that there are n of these quantities.

Now, since x may be taken so near a as to differ from it by an amount as small as we please, we may take x so that

$$u<\frac{k}{n},$$

or $$nu<k,$$

however small k may be.

Then $u+v+w+ \ldots <nu$ (since u is the largest of the quantities u, v, w, ...). Hence $u+v+w+ \ldots <k$, however small k may be; that is, $x+y+z+ \ldots$ may be made to differ fram $a+b+c \ldots$ by an amount as small as we please. Hence

$$\text{limit } (x+y+z+ \ldots)=a+b+c+ \ldots$$

9. THEOREM. *The limit of a constant multiple of a variable equals that constant multiplied by the limit of the variable.*

Let x be a variable and a its limit. We are to prove $\lim nx = na$.

Let $a - x = u$. Then x may be taken so near to a as to make

$$u < \frac{k}{n},$$

or

$$nu < k,$$

however small k may be.

$$a - x = u \therefore na - nx = nu,$$

hence nx may be made to differ from na by an amount as small as we please. Yet nx can never equal na, else x could equal a.

Hence $$\lim nx = na.$$

10. THEOREM. *The limit of the product of two variables equals the product of their limits.*

With the same notation as in Art. 7 we are to prove that

$$\lim xy = ab.$$

$$xy = (a-u)(b-v) = ab - av - bu + uv$$
$$= ab - (av + bu - uv).$$

Since $$\lim v = 0, \lim av = 0,$$

and as $$\lim u = 0, \lim bu = 0,$$

and since u and v are each as small as we please and the product smaller than either, $\lim uv = 0$. And since the limit of each term of $av + bu - uv$ is zero, the limit of the algebraic sum of all three terms is zero. Hence xy may be made to differ from ab by an amount as small as we please; hence $\lim xy = ab$.

11. THEOREM. *The limit of the product of any number of variables is equal to the product of their limits.*

With the same notation as before we are to prove

$$\lim (xyz \ldots) = abc \ldots$$

We have already proved that

$$\lim xy = ab,$$

and we may consider xy a single variable and ab its limit; then by the last article

$$\lim (xy \cdot z) = ab \cdot c,$$

or $$\lim xyz = abc,$$

and now xyz may be considered a single variable and abc its limit, and a repetition of the application of the theorem of the last article would show that the limit of the product of *four* variables equals the product of their limits, and evidently this reasoning could be carried as far as we wish.

12. THEOREM. *The limit of the quotient of two variables equals the quotient of their limits.*

With the same notation as before, we are to prove that

$$\lim \frac{x}{y} = \frac{a}{b}.$$

Let $\frac{x}{y} = q$, then $x = qy$.

$$\therefore \lim x = \lim (qy) = \lim q \cdot \lim y;$$

$$\therefore \lim q = \frac{\lim x}{\lim y} = \frac{a}{b}.$$

13. THEOREM. *The limit of the reciprocal of a variable equals the reciprocal of its limit.*

With the same notation as before, we are to prove that

$$\lim \left(\frac{1}{x}\right) = \frac{1}{a}.$$

We know that $\frac{1}{x} x^2 = x,$

hence $\lim \left(\frac{1}{x} x^2\right) = \lim x = a.$

But $\lim \left(\frac{1}{x} x^2\right) = \lim \left(\frac{1}{x}\right) . \lim x^2$

$$= a^2 . \lim \left(\frac{1}{x}\right),$$

hence $a^2 . \lim \left(\frac{1}{x}\right) = a,$

hence $\lim \left(\frac{1}{x}\right) = \frac{1}{a}.$

14. THEOREM. *The limit of any power of a variable equals that power of the limit of the variable.*

With the same notation as before, we are to prove

$$\lim x^n = a^n,$$

n being any commensurable number either positive or negative, integral or fractional.

First. When n is a positive integer. If in Art. 11 we let y, z, etc., each equal x then b, c, etc., will each equal a, and hence

$$\lim(xxx\ .\ .\ .\ )=aaa\ .\ .\ .$$

or

$$\lim x^n=a^n.$$

Second. When n is a positive fraction, say $\frac{p}{q}$.

Let $x^{\frac{1}{q}}=y$, (1)

then $x=y^q$, (2)

hence by Art. 7 $a=b^q$, (3)

where b is the limit of y; hence

$$b=a^{\frac{1}{q}}. \quad (4)$$

From (1) $x^{\frac{p}{q}}=y^p$, (5)

hence by Art. 7 $\lim x^{\frac{p}{q}}=\lim y^p=b^p$. (6)

But from (4) $b^p=a^{\frac{p}{q}}$,

hence $\lim x^{\frac{p}{q}}=a^{\frac{p}{q}}$;

therefore the theorem is true for any positive exponent whether integral or fractional.

Third. Let n be a negative quantity either integral or fractional, say $n=-s$, then $x^{-s}=\frac{1}{x^s}$; therefore

$$\lim x^{-s}=\frac{1}{a^s}=a^{-s},$$

hence the theorem is true for any commensurable exponents.

INCOMMENSURABLE POWERS.

15. In Chapter II we have found that whatever commensurable numbers are represented by n and r then

$$a^n\,.\,a^r=a^{n+r} \quad (a)$$

$$(a^n)^r=a^{nr} \quad (b)$$

$$a^n\div a^r=a^{n-r} \quad (c)$$

but no meaning has yet been given to quantities with incommensurable indices.

The quantity raised to a power is called the *Base*. In Chapter II, the base was either positive or negative, but the present discussion is confined to the case where the base is positive.

A power of a given base may have more than one value, as for instance, $(25)^{\frac{1}{2}} = \pm 5$, but *any commensurable power of a base has among its values one which is positive.*

For any integral power of a positive base is *evidently* positive, and any *root* of a positive base has among its values one which is positive, and since any *power* of this root is positive so any positive or negative *fractional* power of a positive base has among its values one which is positive, and as a negative power of a base is the reciprocal of a positive power of the same base, any *negative fractional* power of a positive base has among its values one which is positive. This positive value is all that is considered in the present discussion. So that whenever we deal with a quantity like a^x in the present chapter, both a and a^x are positive. These restrictions must not be lost sight of.

16. THEOREM. *If x and y are any two commensurable numbers where y is greater than x, then a^y is greater than a^x if a is greater than unity, and a^y is less than a^x if a is less than unity.*

First Case. When a is greater than unity.

$$a^y \div a^x = a^{y-x}$$

and since $y > x$, $y - x$ is positive, and therefore a^{y-x} is greater than unity, for a positive power or root of a quantity greater than unity is itself greater than unity.

Hence, $\qquad a^y \div a^x > 1$

hence $\qquad a^y > a^x$.

Second Case. Where a is less than unity.

As before $\qquad a^y \div a^x = a^{y-x}$

and $a^{y-x} < 1$, for a positive power or root of a quantity less than unity is itself less than unity.

Therefore $\qquad a^y \div a^x < 1$

and hence $\qquad a^y < a^x$.

Therefore if a is greater than unity, the greater x is the greater is a^x, or in other words, if a is greater than unity, a^x increases as x increases, and if a is less than unity, a^x decreases as x increases.

17. Consider a quantity q intermediate in value between x and y, then if a is greater than unity $a^x < a^q < a^y$, and if a is less

than unity $a^x > a^q > a^y$, so whether a is greater or less than unity, a^q is intermediate between a^x and a^y.

Now consider x and y variables, but always commensurable, and let x increase and y decrease, and suppose them to approach the same incommensurable limit n. As x and y are commensurable, a^x and a^y have definite meanings, and as x and y approach equality, (one increasing and the other decreasing), a^x and a^y also approach equality, or in other words there is some quantity between a^x and a^y from which each of these quantities may be made to differ by an amount as small as we please.

But a^x and a^y can never become equal, since x and y cannot become equal, hence each of these quantities approaches the same *limit.*

Since we have now *proved* that both a^x and a^y *approach a limit,* as x and y themselves approach a limit, we may if we choose neglect y and a^y and fix our attention upon x and a^x remembering, however, that x varies just as it varied before, and hence just as before a^x approaches a limit and indeed the *same* limit.

This limit we will represent by a^n. Thus we have a meaning for a^n where n is incommensurable, viz: it is the limit approached by a^x (x being commensurable) as x approaches a limit n.

18. EXTENSION OF FORMULA (a) OF CHAPTER II TO INCOMMENSURABLE INDICES.

So long as x and y are commensurable we know that

$$a^x a^y = a^{x+y} \qquad (1)$$

Let x approach an incommensurable limit n, and y approach an incommensurable limit r.

Then $x+y$ approaches a limit $n+r$ which is *usually* incommensurable, but may *possibly* be commensurable. Also a^x approaches a limit a^n, a^y approaches a limit a^r and a^{x+y} approaches a limit a^{n+r}.

Now in equation (1) the left-hand member is one variable, and the right-hand member is another which is equal to the first. Hence, Art. 7, $\lim a^x a^y = \lim a^{x+y}$.

But the left-hand member is the product of two variables, hence by Art. 10, $\lim (a^x a^y) = \lim a^x \lim a^y$ or $\lim (a^x a^y) = a^n a^r$, hence $a^n a^r = a^{n+r}$.

19. EXTENSION OF FORMULA *(b)* OF CHAPTER II TO INCOMMENSURABLE INDICES.

With the same notation as in the previous article we have

$$(a^x)^y = a^{xy} \qquad (1)$$

hence $$\lim (a^x)^y = \lim a^{xy} \qquad (2)$$

and by Art. 7 $$\lim a^{xy} = a^{nr} \qquad (3)$$

Let $x = n + u$ and as x may be made to differ from its limit n by an amount as small as we please, u may be made as small as we please, or the limit of u is zero.

Now, by Art. 17, $(a^x)^y = (a^{n+u})^y = (a^n a^u)^y$ (4)

and because y is commensurable

$$(a^n a^u)^y = (a^n)^y (a^u)^y = (a^n)^y a^{uy}. \qquad (5)$$

Substitute in (4) and we get

$$(a^x)^y = (a^n)^y a^{uy}, \qquad (6)$$

hence $$\lim (a^x)^y = \lim [(a^n)^y a^{uy}], \text{ Art. 7}, \qquad (7)$$

and $$\lim [(a^n)^y a^{uy}] = \lim (a^n)^y \lim a^{uy}, \text{ Art. 10} \qquad (8)$$

therefore $$\lim (a^x)^y = \lim (a^n)^y \lim a^{uy}. \qquad (9)$$

But since y approaches r, therefore

$$\lim (a^n)^y = (a^n)^r, \qquad (10)$$

and because u approaches 0 and y approaches r, and hence uy approaches 0, therefore

$$\lim a^{uy} = a^0 = 1. \qquad (11)$$

Substitute for the right-hand member of (9) the values found in (10) and (11) and we get

$$\lim (a^x)^y = (a^n)^r. \qquad (12)$$

Substitute for the two sides of equation (2) the values found in (12) and (3) respectively and we obtain

$$(a^n)^r = a^{nr}.$$

20. EXTENSION OF FORMULA *(c)* OF CHAPTER II, TO INCOMMENSURABLE INDICES.

With the same notation as in the two previous articles we have

$$a^x \div a^y = a^{x-y} \qquad (1)$$

Since $$\lim x = n \text{ and } \lim y = r,$$

$$\lim (x - y) = n - r$$

and $$\lim a^x = a^n, \ \lim a^y = a^r$$

and $$\lim a^{x-y} = a^{n-r}.$$

From (1) $\lim (a^x \div a^y) = \lim a^{x-y}$ by Art. 7.

But $\lim (a^x \div a^y) = \lim a^x \div \lim a^y = a^n \div a^r$,

and $\lim a^{x-y} = a^{n-r}$,

therefore $a^n \div a^r = a^{n-r}$.

21. It is also easy to see that where n is incommensurable

$$a^n b^n = (ab)^n$$

and
$$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$$

For let a and b be two bases and x a variable which remains commensurable, but approaches an incommensurable limit n, then

$$a^x b^x = (ab)^x.$$

Hence $\lim a^x b^x = \lim (ab)^x$ by Art. 7.

But $\lim a^x b^x = \lim a^x \lim b^x = a^n b^n$,

and $\lim (ab)^x = (al)^n$,

hence $a^n b^n = (ab)^n$.

Again
$$\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$$

hence
$$\lim \left(\frac{a^x}{b^x}\right) = \frac{\lim a^x}{\lim b^x} = \frac{a^n}{b^n}$$

and
$$\lim \left(\frac{a}{x}\right)^x = \left(\frac{a}{b}\right)^n$$

hence
$$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$$

22. EXAMPLES OF LIMITS. In the following examples an expression like $\underset{h \succ a}{\text{limit}} \left\{\frac{a-h}{a+h}\right\}$ is to be read: *the limit of* $\left\{\frac{a-h}{a+h}\right\}$ *as h approaches a as a limit.* The symbol $\succ$ stands for the word *approaches.*

1. Find $\underset{h \succ 0}{\text{limit}} \left\{\frac{(x+h)^2 - x^2}{h}\right\}$

Process.

$$\underset{h \succ 0}{\text{limit}} \left\{\frac{(x+h)^2 - x^2}{h}\right\} = \underset{h \succ 0}{\text{limit}} \left\{\frac{x^2 + 2hx - h^2 - x^2}{h}\right\}$$

$$= \underset{h \succ 0}{\text{limit}} \left\{2x + h\right\} = 2x.$$

2. Find $\displaystyle \lim_{x \gtrdot 0} \left\{ \frac{mx}{px^2 - ax} \right\}$

3. Find $\displaystyle \lim_{x \gtrdot a} \left\{ \frac{a^2 - x^2}{a - x} \right\}$

4. Find $\displaystyle \lim_{h \gtrdot 0} \left\{ \frac{(x+h)^3 - x^3}{h} \right\}$.

5. Find $\displaystyle \lim_{x \gtrdot 1} \left\{ \frac{x^3 + 1}{x^2 - 1} \right\}$

6. Prove $\displaystyle \lim_{x \gtrdot a} \left\{ \frac{x^n - a^n}{x - a} \right\} = na^{n-1}$.

23. Limit of the Sum of a Decreasing Geometrical Progression as n Increases. It was noticed in Chapter VIII that if the ratio of a geometrical progression is less than unity, each term of the series is necessarily less than the one preceding it. In this case the series is called a *decreasing* progression.

In the case of a decreasing geometrical progression, it is a little better to write the expression for the sum of the series in the form:

$$s = \frac{a(1 - r^n)}{1 - r}$$

Now if we like we may consider n a variable, and then the two sides of this equation are two variables that are always equal. Therefore, their limits are equal. Whence we may write

$$\lim s \text{ as } n \text{ increases} = \lim \left\{ \frac{a(1 - r^n)}{1 - r} \right\} \text{as } n \text{ increases.}$$

Now since r is less than 1, the term r^n continually approaches the limit 0 as n increases. Whence taking the limit of the right hand member of the equation, we may write:

$$\lim s \text{ as } n \text{ increases} = \frac{a}{1 - r}.$$

24. Examples.

1. Find the limit of the progression (.3333+) as n increases.

Here $a = \frac{3}{10}$ and $r = \frac{1}{10}$.

Whence, $\lim s = \dfrac{\frac{3}{10}}{1 - \frac{1}{10}} = \frac{1}{3}$.

Therefore, $\lim .3333+ = \frac{1}{3}$.

2. Find the limit of the progression $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+$ etc., as n increases.

3. Find the limit of $.272727+$ as n increases.

4. Find the limit of $.279279279+$ as n increases.

5. Find the limit of the sum of $\frac{1}{3}-\frac{1}{6}+\frac{1}{12}-\frac{1}{24}+$ etc., as n increases.

6. Find the limit of the sum of $\sqrt{8}+\sqrt{4}+\sqrt{2}+\sqrt{1}+$ as n increases.

25. THEOREM. *The limit of the sum of the series* $1+r+r^2+r^3+r^4+$ *etc., as r decreases and n increases is* 1.

In the equation

$$\lim s=\frac{a}{1-r}$$

the expression $\frac{a}{1-r}$ will of course have different values for different values of r. Hence we may, if we choose, look upon this expression as a variable. But as r approaches 0 as a limit the fraction $\frac{a}{1-r}$ approaches a as a limit. Therefore we may say that in a decreasing geometrical progression as the number of terms increases without limit, and as the ratio approaches zero as a limit, the sum approaches a as a limit.

In particular, then, if $a=1$ and if the number of terms increases without limit, and the ratio approaches zero as a limit, the series

$$1+r+r^2+\ .\ .\ .$$

approaches 1 as a limit.

26. THEOREM. *The limit of the series*

$$A_0+A_1x+A_2x^2+A_3x^3+\ .\ .\ .\ .,$$

as the number of terms increases without limit and as x approaches zero, is A_0.

Take the series first without the A_0, and suppose K to be a positive quantity numerically equal to the greatest of the coefficients A_1, A_2, A_3, . . . Then

$$A_1x+A_2x^2+A_3x^3+\ .\ .\ .\ \text{numerically} < K(x+x^2+x^3+\ .\ .\ )$$

By Art. 25 the limit of $1+x+x^2+\ldots$, as the number of terms increases and as x approaches zero, equals 1. Therefore,

$$\lim\,(x+x^2+x^3+\ldots)=0$$

and consequently

$$\lim\,K(x+x^2+x^3+\ldots)=0.$$

That is to say, the right member of the inequality above can be made as near zero as we please. Therefore since the left member of the inequality is always numerically less than the right member, the left member can be made to approach zero as near as we please. Hence,

$$\lim\,(A_1x+A_2x^2+A_3x^3+\ldots)=0.$$

That is $\lim\,(A_0+A_1x+A_2x^2+A_3x^3+\ldots)=A_0.$

CHAPTER XII.

UNDETERMINED COEFFICENTS.

1. We know that $\dfrac{x^2-a^2}{x-a}=x+a$, and if we integralize this we obtain an equation of the second degree, but an equation of a different kind from those treated in Chapters IV and V, for the equations previously treated under the name quadratics were shown in Chapter V, Art. 9 to have two roots, and only two; that is, it was shown that there were two and only two values of the unknown quantity which would satisfy the equation; but here we have an equation of the second degree which can be satisfied by any value whatever of x.

The reason is that when the equation is in the integral form we have exactly the same function of x on each side of the sign of equality.

2. THEOREM. *If two functions of x of the n th degree, $A_0+A_1x+\ .\ .\ .\ +A_nx^n$ and $B_0+B_1x+\ .\ .\ .\ +B_nx^n$, are equal for every value of x, then the coefficients of like powers of x on the two sides of the sign of equality are equal each to each.*

If the two functions are equal for every value of x, we have

$$A_0+A_1x+\ .\ .\ .\ +A_nx^n=B_0+B_1x+\ .\ .\ .\ +B_nx^n, \qquad (1)$$

and since this equation is true for any value of x, we may consider x as a variable, varying in any way we please.

Then if we consider x to approach a limit, each side of the equation is a variable which approaches a limit, and we have two variables which are always equal, and each approaches a limit, hence by Chapter XI, Art. 7 the limits are equal. Suppose x to approach zero as a limit then

limit of $A_0+A_1x+\ .\ .\ .\ +A_nx^n=A_0$

and limit of $B_0+B_1x+\ .\ .\ .\ +B_nx^n=B_0$,

hence $A_0=B_0$ by Chap. XI, Art. 7. $\qquad (2)$

Subtracting A_0 from the left side and B_0 from the right side of (1) we get

$$A_1x+A_2x^2+\ .\ .\ .\ +A_nx^n=B_1x+B_2x^2+\ .\ .\ .\ +B_nx^n \qquad (3)$$

Divide (3) by x and we have

$$A_1+A_2x+\ .\ .\ .\ +A_nx^{n-2}=B_1+B_2x+\ .\ .\ .\ +B_nx^{n-1} \qquad (4)$$

Again let x approach zero as a limit, then

$$\text{limit of } A_1+A_2x+ \ . \ . \ . \ +A_nx^{n-1}=A_1,$$
$$\text{limit of } B_1+B_2x+ \ . \ . \ . \ +B_nx^{n-1}=B_1,$$

therefore $A_1=B_1$ by Chap. XI, Art. 7. (5)

Subtracting A_1 from the left side and B_1 from the right side of (4) we get

$$A_2x+A_3x^2+ \ . \ . \ . \ +A_nx^{n-1}=B_2x+B_3x^2+ \ . \ . \ . \ +B_nx^{n-1} \quad (6)$$

Divide (6) by x and we have

$$A_2+A_3x+ \ . \ . \ . \ +A_nx^{n-2}=B_2+B_3x+ \ . \ . \ . \ +B_nx^{n-2}. \quad (7)$$

Then in same way as in the two preceding instances it follows that

$$A_2=B_2$$

and by continuing the process we get

$$A_3=B_3$$
$$A_4=B_4$$
etc.

Therefore if the two functions are equal for all values of x, the coefficients of like powers of x in the two functions are equal each to each.

3. Equations of the kind just considered, which are satisfied by *any* value of x are often called *Identical equations*, while those with which algebra has most to do, those satisfied by *particular* values of x equal in number to the degree of the equation, are often called *Conditional equations.*

4. DEFINITIONS. A *Series* is a succession of terms each of which is derived from one or more of the preceding ones by a fixed law. An *Infinite Series* is one in which the number of terms is unlimited.

An infinite series is *Convergent* if the sum of the first n terms approaches a limit as n increases without limit.

An infinite series is *Divergent* if the sum of the first n terms does not approach a limit as n increases without limit.

The series $1+x+x^2+ \ . \ . \ .$ is convergent if x is less than unity, but divergent if x is equal to or greater than unity.

5. THEOREM. *If for every value of x which makes each of the two series $A_0+A_1x+ \ . \ . \ .$ and $B_0+B_1x+ \ . \ . \ .$ convergent these*

two series approach the SAME *limit as the number of terms increases without limit, then the coefficients of like powers of x in the two series are equal each to each.*

Since we are dealing with the limit of convergent series as the number of terms increases without limit, we know that by taking a sufficient number of terms the sum of the terms taken may be made to differ from the *limit* of the sum by an amount as small as we please.

Let us then write

$$\text{limit } (A_0+A_1x+A_2x^2+ \ . \ . \ . \ +A_{n-1}x^{n-1}+ \ . \ . \ . \ )$$
$$=A_0+A_1x+A_2x^2+ \ . \ . \ . \ +A_{n-1}x^{n-1}+R_1x^n,$$

where R_1x^n is of course the difference between the *limit* of the sum as the number of terms increases without limit and the actual sum of the first n terms.

A_0, A_1, . . . A_{n-1} are each constant, but R_1 is not constant, for if it were the series would terminate. In fact R_1x^n approaches zero as n increases, for if it did not the series would not be convergent. An inspection of the series shows that every term after the first contains x, every term after the second contains x^2, every term after the third contains x^3, and so on; hence every term after the nth will contain the factor x^n, and so it is natural to assume the remainder after n terms are written to be of the *form* R_1x^n.

Instead of writing limit of A_0+A_1x+ . . . as the number of terms increases without limit we write

$$A_0+A_1x+A_2x^2+ \ . \ . \ . \ A_{n-1}x^{n-1}+R_1x^n$$

and in the same way write

$$B_0+B_1x+B_2x^2+ \ . \ . \ . \ +B_{n-1}x^{n-1}+R_2x^n$$

instead of writing limit of $B_0+B_1x+B_2x^2$ . . . as the number of terms increases without limit.

Using this notation we may write

$$A_0+A_1x+ \ . \ . \ . \ +A_{n-1}x^{n-1}+R_1x^n$$
$$=B_0+B_1x+ \ . \ . \ . \ +B_{n-1}x^{n-1}+R_2x^n. \quad (1)$$

If now we consider x as a variable approaching zero we have here two variables which are always equal, and therefore by Chapter IX, Art. 7, their limits are equal. By Chapter IX, Art. 26, the limit of left-hand member equals A_0 and the limit of the right-hand member equals B_0; hence $A_0=B_0$.

Subtract A_0 from the left-hand member and B_0 from the right-hand member of (1) and we get

$$A_1x+A_2x^2+\ \ldots\ +R_1x^n=B_1x+B_2x^2+\ \ldots\ +R_2x^n \qquad (2)$$

Divide both members of (2) by x and we get

$$A_1+A^2x+\ \ldots\ +A_{n-1}x^{n-2}+R_1x^{n-1}$$
$$=B_1+B_2x^2+\ \ldots\ +R_2x^{n-1}. \qquad (3)$$

As before, we have two variables always equal, hence their limits are equal.

But as x approaches zero the limit of the right-hand member equals A_1 and the limit of the left-hand member equals B_1.

Hence, by Chapter XI, Art. 7,

$$A_1=B_1.$$

Repeating the reasoning, we may show successively that

$$A_2=B_2,$$
$$A_3=B_3,$$
etc.

6. The theorem of the last article will enable us to change the form of a function.

The method of doing this consists in assuming a function of the required form with unknown coefficients and then determining the coefficients so that the function assumed shall be identical with the function proposed. The unknown coefficients are determined by placing the proposed function equal to the assumed function, reducing to the rational integral form, and equating the coefficients of like powers of the variables on the two sides of the equation.

If the proposed function can be placed in the assumed form it will be found that there are as many independent compatible equations as there are unknown quantities to determine.

7. DEFINITION. A function is said to be *Developed* or *Expanded* when it is expressed in the form of a series, the sum of whose terms when the number of terms of the series is limited, and the limit of the sum when the number of terms is unlimited, equals the given function.

8. The development of functions is one of the most common applications of the method described in Article 6. The process

is usually referred to as the *method of undetermined coefficients.* We will illustrate the method by working an example.

Let us develope the fraction $\frac{1}{1-x}$.

Assume

$$\frac{1}{1-x}=A_0+A_1x+A_2x^2+\ .\ .\ .\ +A_{n-1}x^{n-1}+Rx^n. \qquad (1)$$

Multiply both sides of (1) by $1-x$ and we get

$$1=A_0+(A_1-A_0)x+(A_2-A_1)x^2+\ .\ .\ .$$
$$+(A_{n-1}-A_{n-2})x^{n-1}+(R-A_{n-1})x^n-Rx^{n-1}.$$

We see that the left hand member contains no power of x except the zero power, or, in other words, in the left hand member, the coefficients of the various powers of x except the zero power are each zero. Hence equating coefficients we get

$$\begin{array}{ll} A_0=1 & \\ A_1-A_0=0\ \therefore & A_1=A_0, \\ A_2-A_1=0\ \therefore & A_2=A_1, \\ A_3-A_2=0\ \therefore & A_3=A_2, \\ \text{etc.,} & \text{etc.} \end{array}$$

From these equations the law of the series is so evident that we can write as many more equations as we please without further calculation.

We thus see from the second column of equations that each coefficient equals the preceding one, and as the coefficient of x^0, or the absolute term, equals 1; therefore each of the other coefficients equals 1. Hence we obtain

$$\frac{1}{1-x}=1+x+x^2+x^3+x^4+\ .\ .\ .$$

As we usually determine only a few of the coefficients, and then discover if we can the law of the series, so it is usual in the assumed series with undetermined coefficients to write only a few terms and indicate the others including the remainder by dots thus:

$$\frac{1}{1-x}=A_0+A_1x+A_2x^2+A_3x^3+\ .\ .\ .$$

Instead of using the method of undetermined coefficients we might have proceeded by ordinary long division as follows:

$$
\begin{array}{llll}
1-x & 1 & & 1+x+x^2+x^3+x^4 \\
 & 1-x & & \\
 & & x & \\
 & & x-x^2 & \\
 & & & x^2 \\
 & & & x^2-x^3 \\
 & & & \quad x^3 \\
 & & & \quad x^3-x^4 \\
 & & & \qquad x^4.
\end{array}
$$

Here as before we obtain

$$\frac{1}{1-x}=1+x+x^2+x^3+x^4+\ .\ .\ .$$

This series on the right side of the sign of equality is convergent if $x<1$, but not otherwise, and therefore this series cannot be called the development of $\frac{1}{1-x}$ unless x is less than unity. See Art. 7. When x is equal to or greater than unity the fraction $\frac{1}{1-x}$ cannot be developed.

9. Examples. Develope the following fractions both by the method of undetermined coefficients and by actual division, and in each case discover the law of the series.

Also in each case state for what values of x the series is a true development.

1. $\dfrac{1}{1+x}$.

2. $\dfrac{1+x}{1-x}$.

3. $\dfrac{a}{a-x}$.

4. $\dfrac{2+3x}{1+x}$.

5. $\dfrac{1}{1-2x+3x^2}$.

6. $\dfrac{1+x}{1-x+x^2}$.

7. $\dfrac{1+2x+3x^2}{1-2x+3x^2}$.

8. $\dfrac{1-2x+3x^2}{1+2x+3x^2}$.

9. $\dfrac{5x+7}{3-4x^2+2x^3}$.

10. $\dfrac{8x^2-3x^3}{3-4x^2+2x^3}$.

Compare the laws of the series in the developments of the fractions in examples 1 and 4; also compare the laws of the series in examples 5 and 7; also in examples 9 and 10.

QUERY: What controls the law of the series in the development of a fraction?

QUERY: How does the numerator affect the development of a fraction in the form of a series?

QUERY: What would the results to examples 7 and 8 suggest about the development of fractions which are reciprocals?

10. It sometimes happens when we try to develope a fraction by the method explained that some of the equations are absurd or contradict one another.

The reason of this is because the fraction cannot be developed into a series *of the form assumed.* Thus if we try to develope $\frac{1}{x-x^2}$ we assume

$$\frac{1}{x-x^2}=A_0+A_1x+A_2x^2+A_3x^3+A_4x^4+\ .\ .\ .$$

Multiply by $x-x^2$ and we get

$$1=A_0x(A_1A_0)x^2+(A_2A_1)x^3+\ .\ .\ .$$

hence

$$1=0,$$
$$A_0=0,$$
$$A_1=0,$$
$$A_2=0,$$
$$\text{etc.}$$

But the first of these equations is false, so we consider that the function cannot be developed into a series of the assumed form.

But we note the denominator of the given fraction contains a factor x, and that hence the fraction proposed equals $\frac{1}{x}\cdot\frac{1}{1-x}$, the second factor of which has already been developed.

From this we would infer that the development of $\frac{1}{x-x^2}$ could be obtained from the development of $\frac{1}{1-x}$ by dividing every term in that development by x.

Hence $$\frac{1}{x-x^2}=x^{-1}+1+x+x^2+x^3+\ldots$$

and we would obtain this very result if should assume

$$\frac{1}{x-x^2}=A_0x^{-1}+A_1+A_2x+A_3x^2+\ldots$$

or in other words if we *begin* the *assumed* series with a term containing x^{-1} instead of beginning with an absolute term. If the fraction we wish to develope is in its lowest terms, and if the lowest power of x that appears in the denominator is the rth power then we must begin our *assumed* series with a term containing x^{-r}.

This is a *safe* rule whether the fraction is in its lowest terms or not, but it is not always *necessary* when the fraction is not in its lowest terms.

In any case when we form an equation by putting a given fraction on the left and an assumed series on the right side of the sign of equality, the assumed series must begin with such a power of x that when the equation is integralized the lowest power of x on the right side of the equation will be as low as the lowest power on the left side.

11. EXAMPLES.

1. Develope $\dfrac{1+x+x^2+x^3}{x^2+2x^5}$.

2. Develope $\dfrac{x+3x^2}{x^2+2x^5}$.

12. Not only fractions but some irrational expressions may be developed by the method of undetermined coefficients.

Let us develope $\sqrt{1-x}$.

Assume

$$\sqrt{1-x}=A_0+A_1x+A_2x^2+A_3x^3+A_4x^4+A_5x^5+\ldots$$

Square each side and we get

$$1-x=A_0+2A_0A_1x+(2A_0A_2+A_1^2)x^2+(2A_0A_3+2A_1A_2)x^3$$
$$+(2A_0A_4+2A_1A_3+A_2^2)x^4+(2A_0A_5+2A_1A_4+2A_2A_3)x^5+\ldots$$

Equating coefficients of like powers of x we get

$$A_0 = 1,$$
$$2A_0A_1 = -1,$$
$$2A_0A_2 + A_1^2 = 0,$$
$$2A_0A_3 + 2A_1A_2 = 0,$$
$$2A_0A_4 + 2A_1A_3 + A_2^2 = 0,$$
$$2A_0A_5 + 2A_1A_4 + 2A_2A_3 = 0,$$
$$2A_0A_6 + 2A_1A_5 + 2A_2A_4 + A_3^2 = 0,$$
etc.

From these we get

$$A_0 = 1,$$
$$A_1 = -\frac{1}{2A_0},$$
$$A_2 = -\frac{A_1^2}{2A_0},$$
$$A_3 = -\frac{2A_1A_2}{2A_0},$$
$$A_4 = -\frac{2A_1A_3 + A_2^2}{2A_0},$$
$$A_5 = -\frac{2A_1A_4 + 2A_2A_3}{2A_0},$$
$$A_6 = -\frac{2A_1A_5 + 2A_2A_4 + A_3^2}{2A_0},$$
etc.

From these the law of the series can be seen.

Taking these equations in order, we find the numerical value of the undetermined coefficients to be as follows:

$$A_0 = 1,\quad A_1 = -\tfrac{1}{2},\quad A_2 = -\tfrac{1}{8},\quad A_3 = -\tfrac{1}{16},\quad A_4 = -\tfrac{5}{128},$$
$$A_5 = -\tfrac{7}{256},\quad A_6 = -\tfrac{21}{1024}.$$

Making these substitutions in the assumed development, we obtain

$$\sqrt{1-x} = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \frac{7x^5}{256} - \frac{21x^6}{1024} - \ldots$$

13. EXAMPLES.

1. Develope $\sqrt{1 + 2x - \frac{x^2}{2}}$.

2. Develope $\sqrt{x + x^2}$.

3. Develope $(1+x)^{\frac{3}{2}}$.

14. It is interesting to note that the development of an irrational expression *may* turn out to be a series of a limited number of terms.

Suppose, for example, we wish to develope $\sqrt{1-2x+x^2}$ and do not recognize that $1-2x+x^2$ is a perfect square, then assume as before

$$\sqrt{1-2x+x^2}=A_0+A_1x+A_2x^2+\ .\ .\ .$$

Square both sides and we get

$$1-2x-x^2=A_0+2A_0A_1x+(A_1^2+2A_0A_2)x^2$$
$$+(2A_0A_3+2A_1A_2)x^3+\ .\ .\ .\ .$$

Equating coefficients of like powers of x and we get

$$\begin{aligned} A_0&=1, \\ 2A_0A_1&=-2 \ \therefore\ A_1=-1, \\ A_1^2+2A_0A_2&=1\ \therefore\ A_2=0, \\ 2A_0A_3+2A_1A_2&=0\ \therefore\ A_3=0, \\ A_2^2+2A_0A_4+2A_1A_3&=0\ \therefore\ A_4=0, \\ \text{etc.,} & \qquad \text{etc.,} \end{aligned}$$

and each of the subsequent coefficients will turn out to be zero, hence we get

$$\sqrt{1-2x+x^2}=1-x.$$

15. In developing irrational expressions it sometimes happens that we should *begin* our *assumed* development with some negative power of x.

An inspection of the proposed example will show with what power of x the development should begin; for the assumed series must be such that, when the equation obtained by putting the given function equal to the assumed series is reduced to the rational integral form, then the lowest power of x on the side which contains the undetermined coefficients must be as low as the lowest power on the other side of the equation.

Thus, to develope $\sqrt{1+\frac{1}{x^2}}$ we would begin the assumed series with a term containing x^{-1}, for when this is squared the lowest power of x is x^{-2} and when both sides are multiplied by x^2 to reduce to the integral form *then* the series on the right side of the equation will begin with an absolute term as it should.

16. If we wish to develope the algebraic sum of two or more radicals it is best to develope each one by itself and then find the algebraic sum of the results.

17. EXAMPLES.

1. Develope $\sqrt{1-\frac{3}{x^2}}$.

2. Develope $\sqrt{x+\frac{1}{x^2}}+\sqrt{1+x^2}$.

3. Develope $\sqrt{1+4x+6x^2+6x^3+5x^4+2x^5+x^6}$.

4. Develope $\sqrt{x^2+2x+3+\frac{2}{x}+\frac{1}{x^2}}$.

CHAPTER XIII.

DERIVATIVES.

1. NOTATION. A definition of a function of a quantity was given in I, Art. 1. To designate a function of x we use the notation $f(x)$.

A function of a quantity is denoted by writing the quantity in a parenthesis and writing the letter f or F or some other functional symbol before the parenthesis. *e. g.*

$f(x)$, $F(x)$, $F_1(x)$ denote functions of x,
$f(y)$, $F(y)$, $f_1(y)$ denote functions of y,
$f(x+h)$, $F(x+h)$, $f'(x+h)$ denote functions of $x+h$,
$f(a)$, $F(a)$, $f_n(a)$ denote functions of a.

The student must be careful not to look upon the expression $f(x)$ as meaning f times x. The symbol f as used here is not a quantity at all, but simply an abbreviation for the words *function of.*

It frequently happens that in the same discussion we wish to refer to different functions of x, in which case we use different functional symbols, as $F(x)$, $f(x)$, $f_1(x)$, $f_n(x)$, $F_n(x)$, etc.

It also frequently happens that in the same discussion we wish to refer to the *same* function of *different* quantities, in which case we use the *same* functional symbol *before* the parenthesis but *different* quantities *within* the parenthesis. *e. g.* If $f(x)$ denotes x^2+1 then $f(a)$ denotes a^2+1, $f(z)$ denotes z^2+1, etc., and if $F(x)$ denotes $\sqrt{x+3}$ then $F(y)$ denotes $\sqrt{y+3}$, $F(x+h)$ denotes $\sqrt{x+h+3}$, etc.

A function of two quantities is any expression in which both of the quantities appear.

If we have to deal with a function of two quantities, say x and y, we use the notation $f(x, y)$ or $F(x, y)$, and if, in the same discussion, we wish to speak of two or more functions of x and y, different functional symbols are used, as $f_1(x, y)$, $f_n(x, y)$, etc.

2. In such an expression as $f(x, y)$ the two quantities x and y are entirely unrestricted in value and independent of each other; but if we have an equation like $f(x, y)=0$, then x and y are to some extent restricted; any value may indeed be given to *one* of the quantities but then the equation fixes the value of the *other*, or in other words, either one of the quantities x or y depends upon the other one; *e. g.*, if $f(x, y)$ stands for $x-y+2$ then when this is not put equal to anything there is no relation between x and y. We may let $x=3$ and $y=5$ or 7 or 10 or any other number. But if we put this same function equal to zero, *then* there is some relation between x and y and they are to *some extent* restricted in value. We may let $x=3$, but then $y=5$ and nothing but 5.

3. If the equation $F(x, y)=0$ can be solved for y, we can express y in terms of x, or y can be determined as a function of x. If we thus determine y we have $y=f(x)$.

In this equation, $y=f(x)$, we may look upon x as a variable, and of course if x varies y will also vary. We may consider x to vary in any way we please, but then the equation determines the way in which y varies. For the reason just stated x is called the *independent variable*, and y, which is a function of x, is called the *dependent variable*.

4. In the equation $y=f(x)$ if a value be given to x then y will have some corresponding value, and if x be given another value different from the first one then y will have some value different from the one *it* had at first. Moreover, the amount by which y thus changes in value will depend in some way upon the amount by which x changes, or in other words, there is some relation connecting the change in value of y with the change in value of x. This relation we will examine, and it will be found to be a very important relation in all that follows.

5. Suppose $f(x)$ to stand for $2x+4$, then putting this equal to y we have

$$y=2x+4.$$

Let us now give to x a series of values, say the successive integers from 1 to 10, and in each case compute the corresponding value of y. The results may be expressed in the form

y	6	8	10	12	14	16	18	20	22	24
x	1	2	3	4	5	6	7	8	9	10

where any number in the lower line is one of the values of x and the number immediately above it is the corresponding value of y.

If $x=2$ the corresponding value of y is 8,

and if $x=10$ the corresponding value of y is 24,

and if x be considered to increase from 2 to 10 then at the same time y will increase from 8 to 24, or, starting at $x=2$, if x increase by 8, y will increase by 16, or if the increase of x is 8, the corresponding increase of y is 16.

Still starting at $x=2$, let us increase x by various amounts and determine the corresponding increase of y.

The results may be arranged in the form

increase of y	16	14	12	10	8	6	4	2
increase of x	8	7	6	5	4	3	2	1

We might have started with some other value of x than 2 and have obtained similar results. In every observed case we see that the increase of y is just twice the increase of x, or in every observed case

$$\frac{\text{increase of } y}{\text{increase of } x}=2.$$

It is easy to see that this is necessarily the case whatever the value of x with which we start and whatever the amount by which x is increased, for if x increases by any amount, $2x$ will increase by just twice that amount and the change in the value of x does not affect the 4, therefore $2x+4$, or y, will increase twice as much as x increases, or

$$\frac{\text{increase of } y}{\text{increase of } x}=2.$$

6. NOTATION. In what follows we deal largely with equations formed by putting y equal to a function of x, and as we will make extensive use of the increase in the value of x and the cor-

responding increase in the value of y it is well to have a convenient notation by which these amounts of increase are denoted. So in future we will use Δx to denote the increase in the value of x and Δy to denote the corresponding increase in the value of y.

In this notation the fraction at the end of Art. 5 would be written

$$\frac{\Delta y}{\Delta x}=2.$$

The student is cautioned not to think of Δx as being Δ times x, for the symbol Δ as here used does not stand for a quantity at all, but simply for the words *increase of*.

7. Let us now consider the equation

$$y=x^2+1.$$

In this equation give x the successive integer values from -3 to 7 and compute the corresponding values of y. We may arrange the results as in Art. 5,

y	10	5	2	1	2	5	10	17	26	37	50
x	-3	-2	-1	0	1	2	3	4	5	6	7

If $x=1$ the corresponding value of y is 2, and if $x=7$ the corresponding value of y is 50, and if x be supposed to increase from 1 to 7, at the same time y will increase from 2 to 50, or starting at $x=1$, if x increase by 6 then y will increase by 48, or when $\Delta x=6$, $\Delta y=48$.

Still starting at $x=1$, let us give to Δx various values and determine the corresponding values of Δy.

The results may be arranged in the form

Δy	48	35	24	15	8	3
Δx	6	5	4	3	2	1

Here we have a case where the ratio $\frac{\Delta y}{\Delta x}$ is not always the same as it was in Art. 5, but at one time it is $\frac{48}{6}$, or 8, at another time it is $\frac{35}{5}$, or 7, etc.

As can be seen by the above scheme, the fraction $\frac{\Delta y}{\Delta x}$ takes successively the values 8, 7, 6, 5, 4, 3 as Δx takes the successive values 6, 5, 4, 3, 2, 1.

We now give to x values intermediate between 1 and 2 and

compute the corresponding values of y. The results may be arranged in the form

y	2	2.0000200001	2.00020001	2.002001	2.0201	2.21
x	1	1.00001	1.0001	1.001	1.01	1.1

As before, let us start at $x=1$ and give to Δx various fractional values and determine the corresponding values of Δy.

The results may be arranged as before in the form

Δy	.21	.0201	.002001	.00020001	.0000200001
Δx	.1	.01	.001	.0001	.00001

An examination of this scheme shows that

when $\Delta x=.1$, then $\dfrac{\Delta y}{\Delta x}=\dfrac{.21}{.1}=2.1$

when $\Delta x=.01$, then $\dfrac{\Delta y}{\Delta x}=\dfrac{.0201}{.01}=2.01$

when $\Delta x=.001$, then $\dfrac{\Delta y}{\Delta x}=\dfrac{.002001}{.001}=2.001$

when $\Delta x=.0001$, then $\dfrac{\Delta y}{\Delta x}=\dfrac{.00020001}{.0001}=2.0001$

when $\Delta x=.00001$, then $\dfrac{\Delta y}{\Delta x}=\dfrac{.0000200001}{.00001}=2.00001$

From the first part of the Article it appears that $\dfrac{\Delta y}{\Delta x}$ is a variable, and from what we have just obtained it further appears that as Δx is taken smaller and smaller the fraction $\dfrac{\Delta y}{\Delta x}$ approaches nearer and nearer the value 2, or in other words, it appears that the fraction $\dfrac{\Delta y}{\Delta x}$ approaches 2, *i. e.* 2 times 1, as Δx approaches zero.

In obtaining the result it is to be noticed that we consider x to increase *from the value* 1, but if we let x increase by various amounts, beginning to count the increase in x *from the value* 2, reasoning exactly as we have just done would lead to the conclusion that $\dfrac{\Delta y}{\Delta x}$ approaches 4, *i. e.* 2 times 2, as Δx approaches zero.

Again, if we begin to count the increase in x *from the value* 3 we would be led to the conclusion that $\dfrac{\Delta y}{\Delta x}$ approaches 6, *i. e.* 2 times 3, as Δx approaches zero.

8. In general, if a be taken as the value of x from which we begin to count the increase of x we would judge from analogy that the fraction $\frac{\Delta y}{\Delta x}$ approaches $2a$ as Δx approaches zero, or, using the notation of Chapter XI, Art. 22,

$$\lim_{\Delta x \doteq 0}\left\{\frac{\Delta y}{\Delta x}\right\}=2a.$$

This we will now prove.

Since $$y=x^2+1, \qquad (1)$$
whatever value be assigned to x the equation will enable us to compute the corresponding value of y.

First, let $x=a$ and represent the corresponding value of y by b, then $$b=a^2+1. \qquad (2)$$

Now let $$x=a+\Delta x$$
and represent the corresponding value of y by $b+\Delta y$, then from equation (1) we get
$$b+\Delta y=(a+\Delta x)^2+1, \qquad (3)$$
or simplifying,
$$b+\Delta y=a^2+2a\Delta x+(\Delta x)^2+1. \qquad (4)$$
Subtract (2) from (4) and we get
$$\Delta y=2a\Delta x+(\Delta x)^2. \qquad (5)$$
Divide (5) by Δx and we obtain
$$\frac{\Delta y}{\Delta x}=2a+\Delta x. \qquad (6)$$

As Δx varies, of course the. the two sides of equation (6) are variables, and, indeed, they are two variables that are always equal, and as Δx approaches zero these two variables each approach a limit.

Hence by Chapter XI, Art. 7, their limits must be equal.

Therefore $$\lim_{\Delta x \doteq 0}\left\{\frac{\Delta y}{\Delta x}\right\}=2a.$$

9. DEFINITION. The value of the fraction $\frac{\Delta y}{\Delta x}$ when that fraction is constant, or the limit of the fraction $\frac{\Delta y}{\Delta x}$ as Δx approaches zero when that fraction is a variable, is called the *Derivative of y with respect to x*, and is represented by the notation $D_x y$, where y is a function of x.

10. The general method of finding the derivative of y with respect to x is that used in Art. 8, viz: give to x some value, say a, and find the corresponding value of y, then give to x a new value, $a+\Delta x$, and again find the corresponding value of y. Subtract the first of the equations thus obtained from the second and we have the value of Δy.

Divide both sides by Δx and we have the value of $\frac{\Delta y}{\Delta x}$.

Then finally find the limit of this fraction as Δx approaches zero.

11. We will now exemplify the method in a few examples.

First. Find $D_x y$ when $y=4x^2+5$. (1)

Let $x=a$ and represent the corresponding value of y by b and we get $b=4a^2+5$. (2)

Now let $x=a+\Delta x$ and the corresponding value of y will be the value b plus the amount by which y has been increased, or $b+\Delta y$, hence $b+\Delta y=4(a+\Delta x)^2+5$. (3)

Expanding, $b+\Delta y=4a^2+8a\Delta x+4(\Delta x)^2+5$. (4)

Subtract (2) from (4) and we get

$$\Delta y=8a\Delta x+4(\Delta x)^2. \tag{5}$$

Divide (5) by Δx and we obtain

$$\frac{\Delta y}{\Delta x}=8a+4(\Delta x). \tag{6}$$

Taking the limit of each side as Δx approaches zero we get

$$\operatorname*{limit}_{\Delta x \doteq 0}\left\{\frac{\Delta y}{\Delta x}\right\}=8a, \tag{7}$$

or

$$D_x y=8a. \tag{8}$$

Second. Find $D_x y$ when $y=cx^2+e$. (1)

Let $x=a$ and represent the corresponding value of y by b, then

$$b=ca^2+e. \tag{2}$$

Now let $x=a+\Delta x$ and we get

$$b+\Delta y=c(a+\Delta x)^2+e, \tag{3}$$

or expanding,

$$b+\Delta y=ca^2+2ac\Delta x+c(\Delta x)^2+e. \tag{4}$$

Subtract (2) from (4) and we get

$$\Delta y=2ac\Delta x+c(\Delta x)^2. \tag{5}$$

Divide both sides of (5) by Δx and we get

$$\frac{\Delta y}{\Delta x}=2ac+c\Delta x. \qquad (6)$$

Taking the limit of each side as Δx approaches zero and we get

$$\underset{\Delta x > 0}{\text{limit}}\left\{\frac{\Delta y}{\Delta x}\right\}=2ac, \qquad (7)$$

or $$D_x y=2ac. \qquad (8)$$

Third. Find $D_x y$ when $y=cx^2+ex+f.$ (1)

Let $x=a$ and represent the corresponding value of y by b and we get

$$b=ca^2+ea+f. \qquad (2)$$

Now let $x=a+\Delta x$ and we get

$$b+\Delta y=c(a+\Delta x)^2+e(a+\Delta x)+f, \qquad (3)$$

or expanding and arranging,

$$b+\Delta y=ca^2+ea+f+(2ac+e)\Delta x+c(\Delta x)^2. \qquad (4)$$

Subtract (2) from (4) and we get

$$\Delta y=(2ac+e)\Delta x+c(\Delta x)^2. \qquad (5)$$

Divide both sides by Δx and we get

$$\frac{\Delta y}{\Delta x}=2ac+e+c\Delta x. \qquad (6)$$

Taking the limit of each side as Δx approaches zero we get

$$\underset{\Delta x > 0}{\text{limit}}\left\{\frac{\Delta y}{\Delta x}\right\}=2ac+e, \qquad (7)$$

or $$D_x y=2ac+e. \qquad (8)$$

12. In what precedes Δx has always been considered positive, but Δx may be negative, in which case x is increased by a negative quantity, or is really *diminished*, so it may be more proper to call Δx the *change in the value of* x than to call it the amount by which x has been *increased*. Either way of speaking is proper provided we understand that the increase *may* be negative. In any case Δx is the amount that must be added to one value of x to give another value of x, and if the second value is greater than the first the amount added will be negative.

The same remark applies to Δy.

13. EXAMPLES. By the method already explained and illustrated find the derivative of the following expressions, supposing in each case that a is the value of x from which the increase of x is counted :

1. $3x+2$. *5.* cx^3.

2. $3x^2+2x$. *6.* $c(x+1)^2$.

3. $(x+1)(x+2)$. *7.* $\frac{1}{x^2}$.

4. $(x+c)^2$. *8.* $\sqrt{x}$.

14. EXTENSION OF MEANING OF $D_x y$.

In Art. 10 we found that when

$$y=4x^2+5, \quad D_x y=8a$$

when $$y=cx^2+e, \quad D_x y=2ac$$

and when $$y=cx^2+ex+f, \quad D_x y=2ac+e$$

In each case $D_x y$ is of course a *constant* as it should be by the definition in Art. 9, where $D_x y$ is defined to be a *limit*, and the limit of a variable is by definition a *constant*.

In each case here noticed $D_x y$ is a constant whose value depends upon the value a from which we begin to count the increase of x, or, as we may say, $D_x y$ is a function of a, while in Art. 5 $D_x y$ was a constant which does *not* depend upon a.

In any case $D_x y$ is either a function of a or it is independent of a, and when it is a function of a the a is the value from which we begin to count the increase of x.

Now, as we may begin to count the increase from *any* value of x, a is of course *any* value of x, and so we may represent it by x instead of a and relieve $D_x y$ from being a constant, or in other words, wherever $D_x y$ was a function of a by the original definition we regard it now and hereafter as the same function of x that, by the original definition, it was of a.

15. Let us now work out a case that was worked in Art. 11, using x now where a was used before.

Take $$y=4x^2+5. \qquad (1)$$

If we write $x+\Delta x$ in place of x and therefore $y+\Delta y$ in place of y we have

$$y+\Delta y=4(x+\Delta x)^2+5. \qquad (2)$$

Expanding,

$$y+\Delta y=4x^2+8x\Delta x+4(\Delta x)^2+5. \qquad (3)$$

Subtract *(1)* from *(3)* and we obtain

$$\Delta y=8x\Delta x+4(\Delta x)^2. \qquad (4)$$

Divide both sides of *(4)* by Δx and we get

$$\frac{\Delta y}{\Delta x}=8x+4\Delta x. \qquad (5)$$

Taking the limit of each side as Δx approaches zero we get

$$\underset{\Delta x \doteq 0}{\text{limit}}\left\{\frac{\Delta y}{\Delta x}\right\}=8x, \qquad (6)$$

or

$$D_x y=8x. \qquad (7)$$

We notice that the result is exactly the same as equation *(8)* in the first example under Art. 10, except that x appears here where a appeared before.

We will hereafter proceed as we have just done and will usually find $D_x y$ as a function of x, but occasionally, as in Art. 5, $D_x y$ will turn out to be a constant.

16. DERIVATIVE OF A CONSTANT.

Let y=a constant, then as x does not appear in the expression for y, x may be changed at pleasure and the change does not affect y, or Δx may have any value whatever, but Δy is always zero.

Hence

$$\frac{\Delta y}{\Delta x}=0,$$

therefore

$$D_x y=0.$$

17. TO FIND THE DERIVATIVE WITH RESECT TO x OF THE ALGEBRAIC SUM OF TWO FUNCTIONS OF x.

Let one function of x be represented by u and the other by v, and let their sum be represented by y; then

$$y=u+v. \qquad (1)$$

When x is increased by Δx suppose that u is increased by Δu, v is increased by Δv and y is increased by Δy, then after x is increased by Δx we have

$$y+\Delta y=u+\Delta u+v+\Delta v. \qquad (2)$$

Subtract *(1)* from *(2)* and we get

$$\Delta y=\Delta u+\Delta v. \qquad (3)$$

Divide both sides of (3) by Δx and we get

$$\frac{\Delta y}{\Delta x}=\frac{\Delta u}{\Delta x}+\frac{\Delta v}{\Delta x}. \qquad (4)$$

Taking the limit of each side of (4) as Δx approaches zero we get

$$\underset{\Delta x > 0}{\text{limit}}\left\{\frac{\Delta y}{\Delta x}\right\}=\underset{\Delta x > 0}{\text{limit}}\left\{\frac{\Delta u}{\Delta x}\right\}+\underset{\Delta x > 0}{\text{limit}}\left\{\frac{\Delta v}{\Delta x}\right\} \qquad (5)$$

or

$$D_x y=D_x u+D_x v.$$

In the same way if $y=u-v$ we would get

$$D_x y=D_x u-D_x v.$$

The result may be expressed thus:

The derivative of the algebraic sum of two functions of x equals the algebraic sum of their separate derivatives.

18. To find the Derivative with Respect to x of the Algebraic Sum of any Number of Functions of x.

Let there be any number of functions of x represented by u, v, w, etc., and let their sum be represented by y; then we have

$$y=u+v+w+\ .\ .\ . \qquad (1)$$

Increase x by the amount Δx and suppose that u, v, w, etc., are increased by the amounts Δu, Δv, Δw, etc., respectively and y is increased by Δy, then we have after x is thus increased

$$y+\Delta y=u+\Delta u+v+\Delta v+w+\Delta w+\ .\ .\ . \qquad (2)$$

Subtract (1) from (2) and we get

$$\Delta y=\Delta u+\Delta v+\Delta w+\ .\ .\ . \qquad (3)$$

Divide both sides of (3) by Δx and we have

$$\frac{\Delta y}{\Delta x}=\frac{\Delta u}{\Delta x}+\frac{\Delta v}{\Delta x}+\frac{\Delta w}{\Delta x}+\ .\ .\ . \qquad (4)$$

Taking the limit of both sides of (4) as Δx approaches zero we have

$$D_x y=D_x u+D_x v+D_x w+\ .\ .\ . \qquad (5)$$

If some of the signs in (1) had been negative the same process could have been applied and the result would have had negative signs in the same positions as they appeared in the original functions.

The result may be expressed thus:

The derivative of the algebraic sum of several functions of x equals the algebraic sum of their separate derivatives.

19. EXAMPLES.

Find the derivative with respect to x of the following expressions:

1. $2x^3+4x^2+x$.

2. x^3+x^2+x+1.

3. x^3-1.

4. x^2+3x+2.

5. x^2-x+1.

6. x^3+1.

20. TO FIND THE DERIVATIVE WITH RESPECT TO x OF THE PRODUCT OF TWO FUNCTIONS OF x.

Let u and v be the two functions of x, and let y be their product, then we have

$$y=uv. \qquad (1)$$

Now increase x by Δx and suppose the corresponding amounts by which y, u, and v respectively increase are Δy, Δu, and Δv, then we have

$$y+\Delta y=(u+\Delta u)(v+\Delta v). \qquad (2)$$

Expanding *(2)* we get

$$y+\Delta y=uv+u\Delta v+v\Delta u+\Delta u\Delta v, \qquad (3)$$

or

$$y+\Delta y=uv+u\Delta v+(v+\Delta v)\Delta u. \qquad (4)$$

Subtract *(1)* from *(4)* and we get

$$\Delta y=u\Delta v+(v+\Delta v)\Delta u. \qquad (5)$$

Divide both sides of *(5)* by Δx and we get

$$\frac{\Delta y}{\Delta x}=u\frac{\Delta v}{\Delta x}+(v+\Delta v)\frac{\Delta u}{\Delta x}. \qquad (6)$$

Taking the limit of each side of *(6)*, remembering that the last term of the right-hand member is the product of two quantities, hence its limit equals the product of their separate limits, and that Δv approaches zero as Δx approaches zero, hence the limit of $v+\Delta v=v$, we get

$$\underset{\Delta x \geq 0}{\text{limit}}\left\{\frac{\Delta y}{\Delta x}\right\}=u \text{ limit } \frac{\Delta v}{\Delta x}+v \text{ limit } \frac{\Delta u}{\Delta x}, \qquad (7)$$

or

$$D_xy=uD_xv+vD_xu. \qquad (8)$$

21. TO FIND THE DERIVATIVE OF THE PRODUCT OF ANY NUMBER OF FUNCTIONS OF x.

First, take three functions of x, say u, v, and w, and let y be their product, then we have

$$y=uvw. \qquad (1)$$

Let $vw=v'$, then $y=uv'$, and hence

$$D_xy=v'D_xu+uD_xv'. \qquad (2)$$

But

$$D_xv'=wD_xv+vD_xw, \qquad (3)$$

hence by substitution in (2) we have

$$D_xy=vwD_xu+uwD_xv+uvD_xw. \qquad (4)$$

Now if we had any number of functions of x, say $u, v, w, \ldots$ and if we let y be their product we have

$$y=uvwz \ldots \qquad (1)$$

Let the product of all the functions after the first be represented by a single letter, that is, let

$$v'=vwz \ldots$$

then

$$y=uv'.$$

Find D_xy as the product of two functions. Then

$$D_xy=v'D_xu+uD_xv'. \qquad (2)$$

Find D_xv' by letting $v'=vw'$, where w' represents the product $wz \ldots$

Substitute the value thus found in (2). The result will contain one term involving D_xw'.

Find the derivative by considering w' to be the product of *two* factors.

Continue this process until finally we reach the product of the last two factors of the expression with which we started.

The result may be stated thus:

The derivative with respect to x of the product of any number of functions is equal to the sum of all the products obtained by multiplying the derivative of each factor by the product of all the other factors.

If the equation here described be divided through by the product of all the given functions, the result may be represented in quite a convenient form, viz:

$$\frac{D_xy}{y}=\frac{D_xu}{u}+\frac{D_xv}{v}+\frac{D_xw}{w}+\frac{D_xz}{z}+\ldots$$

22. Examples.

Find the derivative with respect to x of the following expressions without performing the multiplications indicated:

1. $(x+1)(x+2)$ compare with example 4, Art. 19.

2. $(x^2-x+1)(x+1)$ compare with example 6, Art. 19.

3. $(x^2+x+1)(x-1)$ compare with example 3, Art, 19.

4. $(x^2+1)(x^2+ax+b)$.

23. TO FIND THE DERIVATIVE WITH RESPECT TO x OF THE QUOTIENT OF TWO FUNCTIONS OF x.

Let u and v be the given functions of x, and let y be their quotient, then we have

$$y=\frac{u}{v}. \qquad (1)$$

From (1), by multiplying by v, we get

$$u=vy, \qquad (2)$$

hence

$$D_x u=vD_x y+yD_x v, \qquad (3)$$

or

$$D_x u=vD_x y+\frac{u}{v}D_x v. \qquad (4)$$

Multiply both sides by v and we get

$$vD_x u=v^2D_x y+uD_x v. \qquad (5)$$

Transposing and dividing by v^2 we get

$$D_x y=\frac{vD_x u-uD_x v}{v^2}. \qquad (6)$$

Expressed in words this is

The derivative of a fraction equals the denominator into the derivative of the numerator minus the numerator into the derivative of the denominator all divided by the square of the denominator.

24. EXAMPLES.

Find the derivative with respect to x of the following expressions:

1. $\dfrac{x^3+1}{x+1}$ compare with example 5, Art. 19.

2. $\dfrac{x^3-6x^2+11x-6}{x-3}$ compare with example 4, Art. 19.

3. $\dfrac{x^2-1}{x^2+1}$.

4. $\dfrac{x+1}{x^2+1}$.

25. TO FIND THE DERIVATIVE WITH RESPECT TO x OF A FUNCTION OF ANOTHER FUNCTION OF x.

Suppose y is some function of z, and z is some function of x, then ultimately y is a function of x, hence it has a derivative with respect to x.

But as y is directly a function of z it has a derivative with respect to z.

Moreover, as z is a function of x it has a derivative with respect to x.

We have identically

$$\frac{\Delta y}{\Delta x}=\frac{\Delta y}{\Delta z}\cdot\frac{\Delta z}{\Delta x}. \qquad (1)$$

Taking the limit of each side as Δx approaches zero, remembering that the limit of the product of two variables equals the product of their limits, we have

$$\underset{\Delta x \doteq 0}{\text{limit}}\left\{\frac{\Delta y}{\Delta x}\right\}=\underset{\Delta x \doteq 0}{\text{limit}}\left\{\frac{\Delta y}{\Delta z}\right\}\cdot\underset{\Delta x \doteq 0}{\text{limit}}\left\{\frac{\Delta z}{\Delta x}\right\} \qquad (2)$$

Now z being a function of x we may write

$$z=f(x).$$

and if x be increased by Δx we have

$$z+\Delta z=f(x+\Delta x),$$

and from this it is evident that, as Δx approaches zero, Δz must also approach zero. Hence

$$\underset{\Delta x \doteq 0}{\text{limit}}\left\{\frac{\Delta y}{\Delta z}\right\}=\underset{\Delta z \doteq 0}{\text{limit}}\left\{\frac{\Delta y}{\Delta z}\right\} \qquad (3)$$

Substitute from (3) in (2) and we have

$$\underset{\Delta x \doteq 0}{\text{limit}}\left\{\frac{\Delta y}{\Delta x}\right\}=\underset{\Delta z \doteq 0}{\text{limit}}\left\{\frac{\Delta y}{\Delta z}\right\}\cdot\underset{\Delta x \doteq 0}{\text{limit}}\left\{\frac{\Delta z}{\Delta x}\right\} \qquad (4)$$

The left-hand member of (4) is $D_x y$; the first factor of the right-hand member is $D_z y$, for it is just the same as the left-hand member except that z everywhere takes the place of x; and the second factor of the right-hand member is $D_x z$.

Hence $$D_x y=D_z y\,.\,D_x z. \qquad (5)$$

If $y=z^2$ and $z=x^2+2$, then

$$D_z y=2z \text{ and } D_x z=2x.$$

Hence by equation (5)

$$D_x y=2z\,.\,2x=4zx=4x(x^2+2)=4x^3+8x.$$

It is easy to see that this result is correct, for in the equation $y=z^2$, substitute the value of z and we have

$$y=(x^2+2)^2=x^4+4x^2+4.$$

Hence $$D_x y=4x^3+8x.$$

26. EXAMPLES.

Find the derivative with respect to x of the following expressions:

1. $(x^2+ax+b)^2$.

2. $(x^2+1)^2+3(x^2+1)$.

3. $(x+a)^2+2(x+a)$.

4. $(2x+3)^2+5(2x+3)+4$.

5. $(x^2+3x+2)^2-5(x^2+3x+2)+5$.

6. $2(x^2-1)^3+4(x^2-1)^2+(x^2-1)$.

27. TO FIND THE DERIVATIVE WITH RESPECT TO x OF ANY POSITIVE INTEGRAL POWER OF x.

Let $$y=x^n. \qquad (1)$$

Give to x the value $x+\Delta x$ and we get

$$y+\Delta y=(x+\Delta x)^n. \qquad (2)$$

Expanding the right-hand member of (2) we get

$$y+\Delta y=x^n+nx^{n-1}\Delta x+\frac{n(n-1)}{\lfloor 2}x^{n-2}(\Delta x)^2+\ .\ .\ .\ +(\Delta x)^n. \qquad (3)$$

Subtract (1) from (3) and we get

$$\Delta y=nx^{n-1}\Delta x+\frac{n(n-1)}{\lfloor 2}x^{n-2}(\Delta x)^2+\ .\ .\ .\ +(\Delta x)^n. \qquad (4)$$

Divide both sides of (4) by Δx and we get

$$\frac{\Delta y}{\Delta x}=nx^{n-1}+\frac{n(n-1)}{\lfloor 2}x^{n-2}\Delta x+\ .\ .\ .\ +(\Delta x)^{n-1}. \qquad (5)$$

Taking the limit of each side as Δx approaches zero we have

$$D_x y=nx^{n-1}. \qquad (6)$$

Reasoning exactly as above we could show that when $y=ax^n$,

$$D_x y=nax^{n-1}.$$

This formula may be expressed in words thus—

The derivative with respect to x of ax^n is found by multiplying the exponent by the coefficient and reducing the exponent 1.

It is to be noticed that this formula applies to the derivative with respect to x of a power of x. Of course any other letter besides x could be used to denote a variable.

Thus, when $y=az^n$, $D_z y=naz^{n-1}$.

But we must be careful not to use this formula to find the derivative with respect to some quantity, of a power of some *other* quantity, or, in other words, in order to be able to use this formula the quantity which is raised to a power must be same as that with respect to which the derivative is taken.

28. To find the Derivative with Respect to x of any Negative Integral Power of x.

Let
$$y=x^{-n}=\frac{1}{x^n}. \qquad (1)$$

$$D_x y=\frac{x^n D_x 1 - 1\, D_x x^n}{x^{2n}} \text{ by Art. 23.} \qquad (2)$$

Simplifying, remembering that the derivative of a constant is zero, we get

$$D_x y=-\frac{n\,x^{n-1}}{x^{2n}}=-nx^{-n-1}. \qquad (3)$$

It may be objected to this method that we have used the formula for the derivative of a fraction whose numerator is 1 when that formula supposed that numerator and denominator were each functions of x.

We may then take
$$y=\frac{x}{x^{n+1}}$$

and *now* use the formula of Art. 23 and we get as before

$$D_x y=-nx^{-n-1}.$$

It easily follows that if

$$y=ax^{-n},$$

then
$$D_x y=-nax^{-n-1}.$$

Here, as in Art. 27, in order to use the formula the quantity raised to a power must be the same as the one with respect to which the derivative is taken.

We may express this formula in words thus—

The derivative with respect to x of ax^{-n} is found by multiplying the exponent by the coefficient and reducing the exponent 1.

29. TO FIND THE DERIVATIVE WITH RESPECT TO x OF A FRACTIONAL POWER OF x.

Let	$z=x^{\frac{p}{q}}$	(1)
and let	$y=z^q=x^p.$	(2)
Then	$D_xy=D_zy.D_xz$, Art. 25.	(3)
But	$D_zy=qz^{q-1}$,	(4)
hence	$D_xy=qz^{q-1}.D_xz.$	(5)
But from (2)	$D_xy=px^{p-1}$,	(6)

hence from (5) and (6)

$$qz^{q-1}.D_xz=px^{p-1}. \qquad (7)$$

Divide by qz^q, which is the same as qx^p, and we get

$$z^{-1}D_xz=\frac{p}{q}x^{-1}. \qquad (8)$$

Multiply the left membei by z and the right member by $x^{\frac{p}{q}}$, which is the equal of z, and we obtain

$$D_xz=\frac{p}{q}x^{\frac{p}{q}-1}$$

The same reasoning would show that if $y=ax^{\frac{p}{q}}$, then

$$D_xy=\frac{ap}{q}x^{\frac{p}{q}-1}$$

Hence, as in the two preceding articles, the result is obtained *by multiplying the exponent by the coefficient and reduciug the exponent* 1.

30. EXAMPLES.

Find the derivative with respect to x of the following expressions:

1. $(x^2-x+1)+2(x^2-x+1)$.

2. $\left[\frac{x^3-1}{x+1}\right]+2\left[\frac{x^3-1}{x+1}\right]$

3. $\sqrt[4]{\frac{x^3-1}{x+1}}$.

4. $\frac{\sqrt[4]{x^3-1}}{\sqrt[4]{x+1}}$.

5. $(x+\sqrt{1-x^2})^n$.

6. $\dfrac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}$.

7. $\left|\dfrac{x}{1+\sqrt{1-x^2}}\right|^n$

8. $\left|a^{\frac{1}{2}}+x^{\frac{1}{2}}\right|\sqrt{a^{\frac{1}{2}}+x^{\frac{1}{2}}}$.

9. $\dfrac{\sqrt{a+x}}{\sqrt{a}+\sqrt{x}}$.

10. $\sqrt{\dfrac{1+x}{1-x}}$.

11. $\sqrt{a+\dfrac{b}{x}+\dfrac{c}{x^2}}$.

12. $(a+x)^n.(b+x)^r$.

CHAPTER XIV.

SERIES.

1. A definition of a series was given in XII, Art. 4, and it was there noticed that infinite series are divided into the two classes of convergent and divergent. Convergent series have definite limits as the number of terms is increased without limit, but from their nature divergent series are wholly indefinite, and hence *it is not safe to use divergent series or to base any reasoning upon them.*

In all that follows, and indeed in all that precedes, it is to be understood wherever infinite series are used that the results hold as long as all the series used or obtained are convergent.

In many cases a series is convergent or divergent according to the value of some letter in the series, and it is always understood in such cases that the letter concerned is limited to those values which make the series convergent, and no inference is to be drawn for any other value.

It would be fortunate if some simple and universal criterion were known whereby we might determine whether any given series is convergent or divergent, but unfortunately no such criterion has been found. There are, however, some cases in which we can determine whether a series is convergent or divergent and we give a few of these.

2. Let the terms of a series be represented by u_1, u_2, u_3, etc., in each case the subscript being the same as the number of the term; and let R_1 be the remainder after the first term, R_2 the remainder after the second term, R_3 the remainder after the third term, etc.; in each case the remainder after any number of terms are taken is represented by R with a subscript equal to the number of terms already taken; and further let the sum of any number of terms be represented by S with a subscript equal to the number of terms taken, *i. e.* the sum of two terms will be represented by S_2, the sum of three terms by S_3, and so on.

3. With the notation just explained, the sum of a series which has a limited number of terms will be represented by S_q+R_q, whether q is 1 or 2 or 3 or any other number not exceeding the whole number of terms of the series.

In an infinite *convergent* series S_n approaches a limit as n increases without limit, and the value of this limit is

$$S_q+R_q,$$

where q is any positive whole number whatever. It is easy to see in this case that $R_n \doteq 0$ as n increases without limit.

In an infinite *divergent* series S_n does not approach any limit neither does R_n approach any limit, and S_q+R_q has no definite value at all.

4. It is evident that a series cannot be convergent unless after a certain number of terms are taken the successive terms grow smaller and smaller, or, in other words, unless $u_n \doteq 0$ as n increases without limit. But while this is necessary it is not sufficient, for a series *may* be divergent and still $u_n \doteq 0$ as n increases without limit.

Take for example the series

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ \ldots$$

where the nth term is $\frac{1}{n}$, which evidently approaches zero as n increases without limit.

If this series be grouped thus:

$$1+\tfrac{1}{2}+(\tfrac{1}{3}+\tfrac{1}{4})+(\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8})$$
$$+(\tfrac{1}{9}+\tfrac{1}{10}+\tfrac{1}{11}+\tfrac{1}{12}+\tfrac{1}{13}+\tfrac{1}{14}+\tfrac{1}{15}+\tfrac{1}{16})+\ \ldots$$

then in no group is the sum less than $\frac{1}{2}$ and as there are an unlimited number of groups, the series evidently does not approach any limit; but increases without limit as the number of terms increases without limit, therefore the series is divergent.

5. THEOREM. *A series, all of whose terms are positive, is divergent if nu_n does not approach zero as n increases without limit.*

Since all the terms are positive, nu_n is positive, and since nu_n does not approach zero, we may take r a quantity so near zero that $nu_n>r$, then $u_n>\frac{r}{n}$.

Similarily, $u_{n+1} > \frac{r}{n+1}$, $u_{n+2} > \frac{r}{n+2}$, etc.

Hence

$$u_n + u_{n+1} + u_{n+2} + \ . \ . \ . > r\left\{\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \ . \ . \ .\right\}$$

But the quantities in the parenthesis form the terms of the series $1 + \frac{1}{2} + \frac{1}{3} + \ . \ . \ .$ after the nth term, and this latter series has been shown to be divergent, or in other words, the quantity in the parenthesis increases without limit; and therefore

$$u_n + u_{n+1} + u_{n+2} + \ . \ . \ .$$

increases without limit; therefore the series is divergent.

6. THEOREM. *If the terms of a series are alternately positive and negative and after a certain number of terms each term is numerically less than the preceding one, and the nth term approaches zero, as n increases without limit, the series is convergent.*

Let the series be

$$u_1 - u_2 + u_3 - u_4 + \ . \ . \ .$$

and let the series be represented by S; then we may write either

$$S = S_q + (u_{q+1} - u_{q+2}) + (u_{q+3} - u_{q+4}) + \ . \ . \ . \qquad (1)$$

or

$$S = S_{q+1} - (u_{q+2} - u_{q+3}) - (u_{q+4} - u_{q+5}) - \ . \ . \ . \qquad (2)$$

After a certain number of terms, say k, each term is less than the preceding one, so if q be larger than k, each parenthesis in (1) and also in (2) is positive, and therefore

from (1) $\qquad S > S_q$,

and from (2) $\qquad S < S_{q+1}$.

Thus we see that S is intermediate in value between the two definite quantities S_q and S_{q+1}, which two quantities differ by u_{q+1}.

Similarly, whatever positive whole number be represented by r, we get

$$S > S_{q+2r}$$

and $\qquad S < S_{q+2r+1}$.

Now $\qquad S_{q+2r} > S_q$

and $\qquad S_{q+2r+1} < S_{q+1}$.

Therefore S is intermediate in value between two quantities, the larger of which grows smaller and the smaller of which grows larger.

Moreover, S_{q+2r+1} and S_{q+2r} differ by u_{q+2r+1}, which approaches zero as r increases; therefore the two quantities between which S is always found approach equality as r increases. Therefore S has a definite value, or, in other words, the series is convergent.

7. THEOREM. *If all the terms of a series are positive and after a certain number of terms each term is less than the one before it and the limit of the nth term is zero; then if the limit of the ratio of the $(n+1)$th term to the nth term is less than 1 the series is convergent.*

If all the terms are positive and after a certain number of terms each term is less than the preceding one, then, anywhere after this certain number of terms, the ratio of any term to the preceding one is positive and less than 1. Now, since each of these ratios is less than 1 and the limit of the ratio is less than 1, we may take some quantity, k, less than 1 but so near 1 that each ratio will be less than k.

Hence
$$\frac{u_{n+1}}{u_n} < k \quad \therefore \quad u_{n+1} < ku_n$$
$$\frac{u_{n+2}}{u_{n+1}} < k \quad \therefore \quad u_{n+2} < ku_{n+1} < k^2u_n$$
$$\frac{u_{n+3}}{u_{n+2}} < k \quad \therefore \quad u_{n+3} < ku_{n+2} < k^3u_n$$
$$\frac{u_{n+4}}{u_{n+3}} < k \quad \therefore \quad u_{n+4} < ku_{n+3} < k^4u_n$$
etc.

Therefore
$$u_{n+1}+u_{n+2}+u_{n+3}+\ldots < u_n(k+k^2+k^3+\ldots)$$
or, adding u_n to each side of the inequality,
$$u_n+u_{n+1}+u_{n+2}+u_{n+3}+\ldots < u_n(1+k+k^2+k^3+\ldots)$$
But when $k<1$
$$1+k+k^2+k^3+\ldots = \frac{1}{1-k} \qquad \text{See XII, Art. 8.}$$
and this is a definite quantity.

Therefore the right-hand side of the last inequality is a definite multiple of u_n, and $u_n > 0$; therefore the right-hand side of the last inequality approaches zero; and as the left side is less than the right side and neither side can be negative, therefore the left-

hand side of the last inequality approaches zero; therefore the remainder after $n-1$ terms approaches zero: therefore the series is convergent.

8. In the theorems of the two preceding articles the student should note the force of the words "*after a certain number of terms.*" The first few terms of a series may not give any indication as to whether the series is convergent or divergent. Take, for example, the series

$$1+2x+3x^2+4x^3+5x^4+\ \ldots$$

where the rth term is rx^{r-1}, and suppose $x=\frac{9}{10}$; then the successive terms grow larger up to the ninth term, which $=\frac{9^9}{10^8}$. The tenth term has the same value as the ninth, but every term after the tenth is less than the preceding one. Moreover, as n increases without limit, the nth term approaches zero and the ratio of the $(n+1)$th term to the nth term equals $\left\{1+\frac{1}{n}\right\}x$, and this evidently approaches x as a limit. Hence the series is convergent.

9. THEOREM. *A series is convergent if the series obtained by making all its terms positive is convergent.*

Let the limit of the sum of the positive terms be represented by U_1, and the limit of the sum of the negative terms be represented by U_2; then the limit of the sum of the series will be U_1-U_2.

Now consider a new series formed from the given series by making all its terms positive; then the limit of the sum of this new series will be U_1+U_2, and as this new series is convergent by hypothesis, U_1+U_2 has a definite value. Again, as U_1 and U_2 are both positive and as their sum has a definite value, therefore each of these quantities U_1 and U_2 has a definite value, therefore their difference, U_1-U_2, has a definite value; therefore the series is convergent.

10. THEOREM. *The series*

$$a_0+a_1x+a_2x^2+a_3x^3+\ \ldots$$

is convergent when $x<1$, unless a_n increases without limit as n increases without limit.

We may consider all the terms positive, for if some were negative we could form a new series all of whose terms were positive and conduct the reasoning upon the new series, and if this new series were convergent the original series would be so by Art. 9.

Since we may consider all the coefficients positive and since a_n does not increase without limit we may take b, a quantity greater than the greatest of the coefficients, then

$$a_0+a_1x+a_2x^2+\ .\ .\ .\ < b+bx+bx^2+\ .\ .\ .$$

But the right-hand side of this inequality equals $\frac{b}{1-x}$, (see XII Art. 8,) *i. e.* the right-hand side has a definite value, therefore the left-hand side also has a definite value; hence the series is convergent.

11. THEOREM. *If we have given a series such that after a certain number of terms each term is less than the corresponding term of some series which is known to be convergent, then the given series is convergent.*

Let the given series be

$$u_1+u_2+u_3+u_4+\ .\ .\ . \qquad (1)$$

and let the series known to be convergent be

$$v_1+v_2+v_3+v_4+\ .\ .\ . \qquad (2)$$

and suppose each term after the rth in (1) to be less than the corresponding term in (2).

Since (2) is convergent,

$$v_{r+1}+v_{r+2}+v_{r+3}+v_{r+4}+\ .\ .\ .$$

approaches a definite limit as the number of terms increases without limit, and since $u_{r+1} < v_{r+1}$, $u_{r+2} < v_{r+2}$, $u_{r+3} > v_{r+3}$, etc., therefore $$u_{r+1}+u_{r+2}+u_{r+3}+u_{r+4}+\ .\ .\ .$$

approaches a definite limit as the number of terms increases without limit. Now as the sum of the first r terms of (1) is a definite quantity and the sum of the terms after the rth approaches a definite limit, it follows that the whole series

$$u_1+u_2+u_3+u_4+\ .\ .\ .$$

approaches a definite limit as the number of terms increases without limit, or in other words, the series is convergent.

12. EXAMPLES. Determine whether the following infinite series are convergent or divergent:

1. $1 + \frac{1}{2} + \frac{1}{3^2} + \frac{1}{4^3} + \ldots$

2. $1 + \frac{1^2}{|\underline{2}} + \frac{2^3}{|\underline{3}} + \frac{3^4}{|\underline{4}} + \ldots$

3. $1 + \frac{x}{2} + \frac{x^2}{3^2} + \frac{x^3}{4^3} + \ldots$ when $x < 1$.

4. $1 + \frac{1^2 x}{|\underline{2}} + \frac{2^3 x^2}{|\underline{3}} + \frac{3^4 x^3}{|\underline{4}} + \ldots$ when $x < 1$.

5. $\frac{1}{x} - \frac{1}{x+a} + \frac{1}{x+2a} - \frac{1}{x+3a} + \ldots$ when x and a are positive.

6. $\frac{x}{1 \cdot 2} + \frac{x^2}{2 \cdot 3} + \frac{x^3}{3 \cdot 4} + \frac{x^4}{4 \cdot 5} + \ldots$ when $x < 1$.

7. $\frac{1}{xy} - \frac{1}{(x+1)(y+1)} + \frac{1}{(x+2)(y+2)} - \frac{1}{(x+3)(y+3)} + \ldots$ when x and y are positive quantities.

8. $\frac{x}{1 \cdot 2} + \frac{x^2}{3 \cdot 4} + \frac{x^3}{5 \cdot 6} + \frac{x^4}{7 \cdot 8} + \ldots$ when $x < 1$.

9. $1 + \frac{2^2}{|\underline{2}} + \frac{3^3}{|\underline{3}} + \frac{4^4}{|\underline{4}} + \ldots$

10. $\sqrt{\frac{1}{2}} + \sqrt{\frac{2}{3}} + \sqrt{\frac{3}{4}} + \sqrt{\frac{4}{5}} + \ldots$

11. $\frac{1}{1^2} + \frac{1^2}{2^2} + \frac{2^2}{3^2} + \frac{3^2}{4^2} + \ldots$

12. $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \ldots$

13. Show that $1 + x + \frac{x^2}{|\underline{2}} + \frac{x^3}{|\underline{3}} + \frac{x^4}{|\underline{4}} + \ldots$ is convergent for all values of x.

14. Show that $1 + \frac{2^x}{|\underline{2}} + \frac{3^x}{|\underline{3}} + \frac{4^x}{4} + \ldots$ is convergent or all values of x.

TAYLOR'S FORMULA.

13. Taylor's formula is a very general one that enables us to obtain the development of a function of a binomial $x+h$ arranged according to positive increasing powers of h. Whether the function be integral or fractional, rational or irrational, it matters not; indeed *any* function of a binomial $x+h$ which is *capable* of being expressed in the form of a series arranged according to positive increasing powers of h can be thus expressed by means of Taylor's formula.

Sometimes the series will be finite and sometimes (indeed usually) infinite, but in case the series is infinite it must be remembered that the series and the function cannot be considered equivalent unless the series is convergent.

Before we can take up Taylor's formula it is necessary to explain what is meant by successive derivatives and to give a theorem not given in the chapter on derivatives. These we now take up.

14. SUCCESSIVE DERIVATIVES. If we represent a function of x by $f(x)$ we may find the derivative with respect to x of this function of x, and, as the result is usually another function of x, we may represent it by $f'(x)$. Again, we may find the derivative with respect to x of $f'(x)$ and may represent this by $f''(x)$.

Thus we see that $f''(x)$ is the derivative with respect to x of the derivative with respect to x of $f(x)$. This is called the *second derivative with respect to x of $f(x)$*, and is represented by the notation $D_x^2 f(x)$.

We may find the derivative with respect to x of $f''(x)$ and represent the result by $f'''(x)$. Thus we say that $f'''(x)$ is the derivative with respect to x of the second derivative with respect to x of $f(x)$. This is called the *third derivative with respect to x of $f(x)$*, and is represented by the notation $D_x^3 f(x)$, and so on.

For example, if we take ax^n as the function of x we start with, then

$$D_x ax^n = nax^{n-1}$$
$$D_x^2 ax^n = D_x nax^{n-1} = n(n-1)ax^{n-2}$$
$$D_x^3 ax^n = D_x n(n-1)ax^{n-2} = n(n-1)(n-2)ax^{n-3}$$

etc.

15. EXAMPLES.

1. Find 5 successive derivatives of x^4.

2. Find 4 successive derivatives of $x^5+x^4+x^3+x^2+x+1$.

3. Find 3 successive derivatives of $\dfrac{1}{1-x}$.

4. Find 3 successive derivatives of $\sqrt{1+x}$.

5. Find 3 successive derivatives of $\sqrt{1+2x}$.

6. Find 3 successive derivatives of $\sqrt{1+x^2}$.

7. Find 3 successive derivatives of $\dfrac{1+x}{1-x}$.

16. THEOREM. *In a function of a binomial $x+h$, say $f(x+h)$, the derivative with respect to x, when h is regarded constant, is equal to the derivative with respect to h when x is regarded constant.*

Let $y=f(x+h)$, and let $x+h=z$, then

$$D_x y = D_z y \,.\, D_x z. \quad \text{See XIII, Art. 25, Eq. (5)}$$

But $$D_x z = 1.$$

Hence $$D_x y = D_z y. \qquad (1)$$

Again, $$D_h y = D_z y \,.\, D_h z. \quad \text{See XIII, Art. 25, Eq. (5)}$$

But $$D_h z = 1.$$

Hence $$D_h y = D_z y. \qquad (2)$$

From (1) and (2) it follows that

$$D_x y = D_h y.$$

17. TAYLOR'S FORMULA. We are now prepared to take up Taylor's formula.

If $f(x+h)$ can be developed into a series arranged according to positive increasing powers of h, let us assume

$$f(x+h)=A_0+A_1h+A_2h^2+A_3h^3+\dots \qquad (1)$$

where A_0, A_1, A_2, etc., do not contain h, but are in general functions of x.

Take the derivative with respect to x of each side of (1) and we have

$$D_x f(x+h)=D_xA_0+h\,D_xA_1+h^2D_xA_2+h^3D_xA_3+\dots \qquad (2)$$

Also take the derivative with respect to h of each side of (1) and we have

$$D_h f(x+h)=A_1+2A_2h+3A_3h^2+4A_4h^3+\dots \qquad (3)$$

By Art. 16 these two expressions must be equal, therefore equating coefficients of like powers in (2) and (3) we get

$$A_1 = D_x A_0 \tag{4}$$

$$2A_2 = D_x A_1 \quad \therefore \quad A_2 = \tfrac{1}{2} D_x A_1 = \frac{1}{2} D_x^2 A_0 \tag{5}$$

$$3A_3 = D_x A_2 \quad \therefore \quad A_3 = \tfrac{1}{3} D_x A_2 = \frac{1}{2\cdot 3} D_x^3 A_0 \tag{6}$$

$$4A_4 = D_x A_3 \quad \therefore \quad A_4 = \tfrac{1}{4} D_x A_3 = \frac{1}{2\cdot 3\cdot 4} D_x^4 A_0 \tag{7}$$

$$5A_5 = D_x A_4 \quad \therefore \quad A_5 = \tfrac{1}{5} D_x A_4 = \frac{1}{2\cdot 3\cdot 4\cdot 5} D_x^5 A_0 \tag{8}$$

etc.

Now if we make $h=0$ in (1) it is easy to see that $A_0 = f(x)$, where $f(x)$ means the same function of x that the given function is of $x+h$, or in other words, $f(x)$ is what the given function becomes when h is put equal to zero.

If, in (1), we substitute $f(x)$ for A_0, and for the coefficients of the various powers of h the values found in equations (4) to (8), we get

$$f(x+h) = f(x) + h\,D_x f(x) + \frac{h^2}{\lfloor 2} D_x^2 f(x) + \frac{h^3}{\lfloor 3} D_x^3 f(x) + \ldots\ldots$$

This result is Taylor's formula and is often spoken of as Taylor's theorem.

18. Application of Taylor's Formula.

Let us develope $(x+h)^6$ by Taylor's formula.

Here $$f(x+h) = (x+h)^6.$$

Therefore $$f(x) = x^6.$$

Finding the successive derivatives of x^6 we get

$$D_x f(x) = 6x^5, \qquad D_x^2 f(x) = 6\,.\,5x^4,$$
$$D_x^3 f(x) = 6.5.4x^3, \qquad D_x^4 f(x) = 6.5.4.3x^2,$$
$$D_x^5 f(x) = 6.5.4.3.2x, \qquad D_x^6 f(x) = 6.5.4.3.2.1,$$
$$D_x^7 f(x) = 0,$$

and every derivative after the seventh will equal zero. Therefore by substitution in Taylor's formula we get

$$(x+h)^6 = x^6 + 6x^5h + \frac{6\,.\,5}{2}x^4h^2 + \frac{6.5.4}{2\,.\,3}x^3h^3$$
$$+ \frac{6.5.4.3}{2.3.4}x^2h^4 + \frac{6.5.4.3.2}{2.3.4.5}xh^5 + \frac{6.5.4.3.2}{2.3.4.5.6}h^6,$$

or $$(x+h)^6 = x^6 + 6x^5h + 15x^4h^2 + 20x^3h^3 + 15x^2h^4 + 6xh^5 + x^6.$$

This result is seen to be the same as that obtained by a direct application of the Binomial formula, which of course is as it ought to be.

As a second example, let us develope $\sqrt{x+h}$ by Taylor's formula.

Here $$f(x+h)=(x+h)^{\frac{1}{2}}.$$

Therefore $$f(x)=x^{\frac{1}{2}}.$$

Finding the successive derivatives of $x^{\frac{1}{2}}$ we get

$$D_x f(x)=\tfrac{1}{2}x^{-\frac{1}{2}}, \qquad D_x^2 f(x)=-\tfrac{1}{2}\cdot\tfrac{1}{2}x^{-\frac{3}{2}},$$
$$D_x^3 f(x)=\tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\tfrac{3}{2}x^{-\frac{5}{2}}, \quad D_x^4 f(x)=-\tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\tfrac{3}{2}\cdot\tfrac{5}{2}x^{-\frac{7}{2}},$$
etc.

Making the substitutions in Taylor's formula we get

$$\sqrt{x+h}=x^{\frac{1}{2}}+\tfrac{1}{2}x^{-\frac{1}{2}}h-\tfrac{1}{8}x^{-\frac{3}{2}}h^2+\tfrac{1}{16}x^{-\frac{5}{2}}h^3-\tfrac{5}{128}x^{-\frac{7}{2}}h^4+\ \ldots$$

If in this equation we make $x=1$ we get

$$\sqrt{1+h}=1+\tfrac{1}{2}h-\tfrac{1}{8}h^2+\tfrac{1}{16}h^3-\tfrac{5}{128}h^4+\ \ldots$$

If in this equation we change the sign of h we get

$$\sqrt{1-h}=1-\tfrac{1}{2}h-\tfrac{1}{8}h^2-\tfrac{1}{16}h^3-\tfrac{5}{128}h^4-\ \ldots$$

Compare this with the result obtained in XII, Art. 12.

It was stated in Art. 13 that Taylor's formula could be used to develope *any* function of a binomial which is capable of being developed into a series arranged according to positive increasing powers of one of the quantities. It is indeed a matter of substitution, but care must be taken that the substitution be such that the development obtained is arranged according to positive increasing powers of the proper quantity.

If, for example, we wish to develope $\sqrt{x+1}$ into a series arranged according to positive increasing powers of x, it might at first appear that, in the development of $\sqrt{x+h}$, we could simply make $h=1$; but this would give us a series arranged according to positive increasing powers of 1, not x. The proper course is as follows: First, develope $\sqrt{x+h}$ according to positive increasing powers of h; then in this result make $x=1$ and we have the development of $\sqrt{1+h}$ arranged according to positive increasing powers of h; then, finally, in *this* result, change h into x and we obtain the result sought.

19. Binomial Theorem for any Exponent.

Let us apply Taylor's formula to the development of $(x+h)^n$ according to positive increasing powers of h, where n is either *positive or negative, integral or fractional.*

Here, then, $\quad f(x+h)=(x+h)^n.$

Therefore $\quad f(x)=x^n.$

Finding the successive derivatives of x^n we obtain

$D_x f(x)=nx^{n-1}, \quad D_x^2 f(x)=n(n-1)x^{n-2},$

$D_x^3 f(x)=n(n-1)(n-2)x^{n-3}, \quad D_x^4 f(x)=n(n-1)(n-2)(n-3)x^{n-4}$

etc.

Therefore by substituting in Taylor's formula we get

$$(x+h)^n=x^n+nx^{n-1}h+\frac{n(n-1)}{\underline{|2}}x^{n-2}h^2+\frac{n(n-1)(n-2)}{\underline{|3}}x^{n-3}h^3$$
$$+\frac{n(n-1)(n-2)(n-3)}{\underline{|4}}x^{n-4}h^4+\ldots \quad (1)$$

Thus we arrive at the Binomial formula, where, however, the exponent is not restricted, as in Chapter X, to being a positive whole number.

From this we see that the Binomial formula in its greatest generality is a special case of Taylor's formula.

The series (1) will be finite if n is a positive whole number, but not otherwise.

When the series (1) is infinite it should be examined to see whether it is convergent or divergent, for values *may* be given to x, h, n, which will render the equation (1) untrue.

For example, let $x=1$, $h=-3$, and $n=-2$; then the left-hand member of (1) becomes $(1-3)^{-2}$, which equals $(-2)^{-2}$, which equals $\frac{1}{(-2)^2}$, which equals $\frac{1}{4}$, a definite quantity.

But the right-hand member of (1) becomes $1+6+27+108+\ldots$ a sum of positive whole numbers each greater than the one before it, and evidently the sum does not approach $\frac{1}{4}$.

20. Examples.

1. Develope $(1-x)^{-1}$ by Taylor's formula.

2. Develope $(1-x)^{-1}$ by the Binomial formula.

Compare *(1)* and *(2)* with XII, Art. 8.

3. Develope $(1+x)^{\frac{3}{2}}$ by Taylor's formula.

4. Develope $(1+x)^{\frac{3}{2}}$ by the Binomial formula.
Compare *(3)* and *(4)* with XII, Art. 13, Ex. 3.

5. Develope $(1-2x)^{\frac{1}{2}}$ by the Binomial formula.

6. Develope $(a-2x)^{-\frac{2}{3}}$ by the Binomial formula.

7. Develope $(c^2+x^2)^{-\frac{1}{2}}$ according to positive increasing powers of x by Taylor's formula.

8. Develope $(c^2+x^2)^{-\frac{1}{2}}$ according to positive increasing powers of c by Taylor's formula.

9. Develope $(c^2+x^2)^{-\frac{1}{2}}$ according to positive increasing powers of x by the Binomial formula.

10. Develope $(c^2+x^2)^{-\frac{1}{2}}$ according to positive increasing powers of c by the Binomial formula.

11. Find the first negative term in the development of $(1+x)^{\frac{10}{3}}$ by the Binomial formula.

CHAPTER XV.

LOGARITHMS.

1. After the extension of the theory of indices in Chapter XI so as to embrace incommensurable exponents, we are enabled to give an interpretation to the expression

$$a^x$$

for all possible values of x, integral or fractional, commensurable or incommensurable. Since x appears in this expression in such an unrestricted form it is common to speak of the expression as an *exponential function of* x, intending to call attention thereby to the fact that x may be considered a continuous variable as in any ordinary algebraic function.

If in the equation

$$a^x = y,$$

we assume x to pass from one extreme of the algebraic scale to the other, taking in every possible value, then we are able to give a meaning to this equation in two variable; because for every possible value of x, a^x, that is, y, has a definite meaning and value.

In this connection it must be remembered that we are using a and a^x under the restrictions mentioned in XI, Art. 15. So that *when we speak of* a^x *we mean that* a *is a positive number, and by the value of* a^x *we mean that one of its values which is positive.* Hence in the equation $a^x = y$ we are to think of but one value of y resulting when any particular value is assigned to x. Thus in $10^{\cdot 25} = y$ we are to understand $y = +\sqrt[4]{10}$ and not $y = -\sqrt[4]{10}$ or any other possible value of y.

Of course the very restrictions just mentioned prevent y from having a *negative* value. Moreover, it is not evident that y can have every positive value we please. For example, is it not plain that a value of x exists which satisfies the equation $10^x = \pi$. In general, while it is easy to see that in the equation

$$a^x = y$$

there always exists a value of y for any value assigned to x, it is far from evident that there exists a value of x corresponding to every value which may be assigned to y. Whence the necessity for the following theorems.

2. THEOREM. *The expression $a^{\frac{1}{x}}$ can be made to differ from* 1 *by less than any assigned quantity if x be sufficiently increased.*

Suppose it be required to increase x so that

$$a^{\frac{1}{x}} < 1 + d, \tag{1}$$

where d stands for an assigned positive number. Then we must have

$$(1+d)^x > a, \tag{2}$$

or, by the binomial theorem,

$$1 + xd + \frac{x(x-1)}{1 \cdot 2} d^2 + \ldots > a. \tag{3}$$

It is easy to see that $1 + xd$ can always be made greater than a, however small d may be. Much more, then, will the left member of (3) be greater than a. In fact, the inequality

$$1 + xd > a$$

will hold if

$$xd > a - 1,$$

or if

$$x > \frac{a-1}{d}.$$

Hence to make $a^{\frac{1}{x}}$ less than $1 + d$ take

$$x > \frac{a-1}{d}.$$

Example: Find x such that $10^{\frac{1}{x}} < 1.0001$.

Here $d = .0001$ and $a = 10$; whence

$$x > \frac{9}{.0001}, \text{ or } 90000.$$

3. THEOREM. *The expression a^x is a continuous function of x.*

Suppose $a^x = y$ and let x take on any increase, s, and suppose the corresponding value of y be $y + t$, so that

$$a^{x+s} = y + t. \tag{1}$$

We are to prove that as x passes *continuously* from x to $x+s$ that y passes *continuously* from y to $y+t$; that is, as x changes from x to $x+s$ *by passing over every intermediate value* that y changes from y to $y+t$ *by passing over every intermediate value.*

The equation $a^{x+s} = y + t$ may be written

$$a^x a^s = y + t, \tag{2}$$

and since $a^x = y$, this may be written

$$a^x a^s = a^x + t, \tag{3}$$

or

$$a^x(a^s - 1) = t. \tag{4}$$

Now, by the last theorem, by taking s small enough a^s may be made to differ from 1 by an amount as small as we please. Hence in equation (4) t may be made as small as we please by taking s small enough. That is, the difference between two successive values of a^x can be made as small as we please. Therefore it is a continuous function of x.

4. It follows directly from the above theorem that *for every positive value which may be assigned to y in the equation $a^x = y$, a corresponding value of x exists which will satisfy the equation.*

For the last article shows that as x is increased continuously from the value 0 without limit that y increases continuously from the value 1 without limit. That is, y may have every value greater than 1. It is also seen that as x is decreased continuously from the value 0 without limit that y decreases continuously from the value 1. That is, y may have every fractional value.

The above shows that if any value be assigned to y in the equation $a^x = y$ that a value of x *exists* which will satisfy it, but it does not explain how to find that value. Thus it does not show how to find x in the equation $10^x = 5$. The method of finding this will be explained later.

5. DEFINITIONS. In the equation $a^x = y$, where a is some chosen positive number not 1;

The constant quantity a is called the *Base.*

The quantity y is called the *Exponential* of x to the base a.

The quantity x is called the *Logarithm* of y to the base a, and is written $x = \log_a y$.

The use of the word logarithm may be kept in mind by remembering this sentence: In the equation $a^x = y$, x is called the *Exponent* of the power of a or the *Logarithm* of y.

Of course the two equations

$$a^x = y \qquad (1)$$

$$x = \log_a y \qquad (2)$$

express the same truth respecting the relation between x and y. The second equation uses the logarithmic notation and is always to be interpreted by means of the first equation.

If in the equation $a^x = y$, where a is some positive number not 1, different values be assigned to y and the corresponding values

of x be computed and tabulated, the results constitute a *System of Logarithms*.

Since any positive value except 1 may be chosen for the base a, the number of different possible systems of logarithms is unlimited. In fact, however, only two systems have ever been tabulated; the *Natural* or *Naperian* or *Hyperbolic System*, whose base is approximately 2.7182818+, and the *Common* or *Briggs' System*, whose base is 10.

The Naperian logarithms of all numbers from 1 to 20,000 have been computed to 27 places of decimals. The common logarithms of all numbers from 1 to over 200,000 have been found. They are usually printed to seven decimal places, but they have been computed to many more.

The great value of a table of logarithms is the immense amount of labor which can be saved by its use in multiplication, division, evolution, or involution of numbers, as will be explained hereafter.

5. EXAMPLES. Write the following equations, using the logarithmic notation:

1. $10^x = \pi$.

2. $e^x = y$.

3. $11^2 = 121$.

4. $10^3 = 1000$.

5. $16^{.25} = 2$.

6. $10^0 = 1$.

7. $10^{-3} = .001$.

8. $10^{.25} = 1.77828+$

9. $a^{n+r} = a^n a^r$.

10. $a^1 = a$.

11. $a^{\log_a y} = y$.

12. $10^{\log_{10} y} = y$.

13. $e^B = a$.

14. $10^{.301030} = 2$.

Express the following, using the exponential notation:

15. $\log_{27}(\frac{1}{3}) = -.3333+$

16. $\log_{10}4 = .602060$

17. $\log_{10}10000 = 4$.

18. $\log_{10}.00001 = -5$.

19. $\log_2 1024 = 10$.

20. $\log_e e = 1$.

21. $\log_b b^b = b$.

22. $\log_e a = B$.

6. PROPERTIES OF LOGARITHMS. Inasmuch as logarithms are merely the exponents of a fixed base, the properties of logar-

ithms are entirely dependent upon the properties of exponents in general, which have already been established.

Among the fundamental properties of logarithms are these:

The logarithm in any system of the base itself is 1.

For $$a^1=a,$$

that is, $$\log_a a=1.$$

The logarithm of unity in all systems is 0.

For $$a^0=1,$$

that is, $$\log_a 1=0.$$

Negative numbers have no logarithms.

For in the equation $a^x=y$, a is positive by supposition and by the value of a^x we mean that one of its values which is positive. Hence y cannot be negative. See Art. 1.

If we understand the same system of logarithms to be used throughout, then the following four theorems hold.

7. THEOREM. *The logarithm of the product of several numbers equals the sum of the logarithms of the separate factors.*

Let n and r be any two positive numbers and let

$$\log_a n=x \text{ and } \log_a r=z. \tag{1}$$

Then, by the definition of a logarithm,

$$n=a^x \text{ and } r=a^z.$$

Multiplying these equations together, member by member,

$$nr=a^{x+z}.$$

That is, $$\log_a nr=x+z,$$

or, from (1), $$\log_a nr=\log_a n+\log_a r. \tag{a}$$

In the same way, if $\log_a s=u$, then

$$nrs=a^{x+z+u}.$$

That is, $$\log_a nrs=\log_a n+\log_a r+\log_a s.$$

8. THEOREM. *The logarithm of the quotient of two numbers equals the logarithm of the dividend minus the logarithm of the divisor.*

Let n and r be any two positive numbers and let

$$\log_a n=x \text{ and } \log_a r=z. \tag{1}$$

Then, by definition, $$n=a^x \text{ and } r=a^z.$$

Consequently $$\frac{n}{r}=\frac{a^x}{a^z}=a^{x-z},$$

Therefore, by definition,

$$\log_a\left[\frac{n}{r}\right]=x-z,$$

or, by equation (1),

$$\log_a\left[\frac{n}{r}\right]=\log_a n-\log_a r. \qquad (b)$$

9. THEOREM. *The logarithm of a power of a number is equal to the logarithm of the number multiplied by the exponent of the power.*

Let n be any number, and let $\log_a n=x$. Then, by definition,

$$n=a^x.$$

Consequently $$n^p=a^{px}.$$

Therefore, by definition, $$\log_a n^p=px.$$

That is, $$\log_a n^p=p \log_a n. \qquad (c)$$

10. THEOREM. *The logarithm of any root of a number is equal to the logarithm of the number divided by the index of the root.*

Let n be any number, and let $\log_a n=x$. Then, by definition,

$$n=a^x.$$

Consequently $$\sqrt[q]{n}=a^{\frac{x}{q}}.$$

Therefore, by definition,

$$\log_a(\sqrt[q]{n})=\frac{x}{q}.$$

That is, $$\log_a(\sqrt[q]{n})=\frac{\log_a n}{q}. \qquad (d)$$

11. THEOREM. *If several numbers are in geometrical progression, their logarithms are in arithmetical progression.*

Let the numbers which are in geometrical progression be represented by

$$n,\ nr,\ nr^2,\ nr^3,\ \ldots$$

Then their logarithms to the base a form the series

$$\log_a n,\ \log_a n+\log_a r,\ \log_a n+2\log_a r,\ \log_a n+3\log_a r,\ \ldots$$

which is an arithmetical progression with the common difference equal to $\log_a r$.

12. EXAMPLES. In these examples and in all the following pages the *Common Logarithm* is designated by the symbol log instead of $\log_{10}$. Hence when no subscript appears we are to understand that the base is 10.

1. $\log(1888 \times 476 \div 1492) = \log 1888 + \log 476 - \log 1492.$

2. $\log[\sqrt{789} \times (\frac{239}{930})^5] = \frac{1}{2}\log 789 + 5\log 239 - 5\log 930.$

3. $\log\left[\frac{\sqrt{2}\,\sqrt[3]{5}}{\sqrt[7]{10}}\right] =$ what?

4. $\log_b(c^2de \div lkm) =$ what?

5. $\log_b(h^5g\sqrt[3]{ad^2}) =$ what?

6. $\log_b\left\{\frac{b}{x}\sqrt{\frac{b}{x}}\right\} =$ what?

7. $\log_b\sqrt[x]{\frac{d^q e}{3k^x}} =$ what?

8. $\log_b b^b =$ what?

9. Prove $\log_a(\log_b b^b) = \log_a b.$

10. Prove $\log_a e = \frac{1}{\log_e a}.$

13. CHARACTERISTIC AND MANTISSA. For reasons which will appear later the common logarithm of a number is always written so that it shall consist of a positive decimal part less than 1 and an integral part which may be either positive or negative. Thus the common logarithm of .0256 is really -1.59176, since

$$10^{-1.59176} = \frac{1}{10^{1.59176}} = .0256.$$

But instead of writing

$$\log .0256 = -1.59176$$

we write the equivalent equation

$$\log .0256 = -2 + .40824,$$

or, as is the universal custom, with the minus sign *over* the 2,

$$\log .0256 = \bar{2}.40824$$

The minus sign over the 2 shows that 2 is alone affected; that is, the decimal fraction following it is positive. *The student must always take especial care to correctly interpret this method of notation.*

When the logarithm of a number is arranged so that it consists of a positive decimal part less than 1 and an integral part either

positive or negative, special names are given to each part. The positive or negative integral part is called the *Characteristic* of the logarithm. The positive decimal part is called the *Mantissa*.

14. The following table is self-explanatory:

$10^4 = 10000$,	whence	$\log 10000 = 4$
$10^3 = 1000$,	"	$\log 1000 = 3$
$10^2 = 100$,	"	$\log 100 = 2$
$10^1 = 10$,	"	$\log 10 = 1$
$10^0 = 1$,	"	$\log 1 = 0$
$10^{-1} = .1$,	"	$\log .1 = -1$
$10^{-2} = .01$,	"	$\log .01 = -2$
$10^{-3} = .001$,	"	$\log .001 = -3$
$10^{-4} = .0001$,	"	$\log .0001 = -4$

Here we observe that as the numbers pass through the series 10000, 1000, 100, 10, etc., the logarithms pass through the series 4, 3, 2, 1, etc.; that is, continuous division of the number by 10 corresponds to a continuous subtraction of 1 from its logarithm. This can easily be shown to hold in any case.

15. THEOREM. *Multiplying any number by* 10 *increases the common logarithm by* 1, *and dividing any number by* 10 *decreases its common logarithm by* 1.

Let y be any number and x its common logarithm. Then

$$\log y = x,$$

or

$$10^x = y.$$

We are to prove $\log 10y = x + 1,$

and $\log \frac{1}{10}y = x - 1.$

By formula *(a)*, Art. 7,

$$\log 10y = \log y + \log 10.$$

But $\log 10 = 1$ (Art. 6) and $\log y = x$.

Hence, substituting, $\log 10\,y = x + 1$.

Also by formula *(b)*, Art. 8,

$$\log \tfrac{1}{10}y = \log y - \log 10.$$

That is, $\log \frac{1}{10}y = x - 1.$

16. COROLLARY. *Moving the decimal point in a number one place to the right increases its common logarithm by* 1, *and moving it one place to the left decreases its logarithm by* 1.

17. COROLLARY. *The common logarithms of all numbers consisting of the same significant figures have the same mantissa.*

For moving the decimal point merely adds or subtracts 1 from the logarithm; that is, merely affects the characteristic. Thus

$$\begin{aligned} \log 256 &= 2.40824 \\ \log 25.6 &= 1.40824 \\ \log 2.56 &= 0.40824 \\ \log .256 &= \bar{1}.40824 \\ \log .0256 &= \bar{2}.40824 \\ \log .00256 &= \bar{3}.40824 \end{aligned}$$

18. THEOREM. *If a number has its first significant figure in units' place, the characteristic of its common logarithm is* 0.

If the number has its first significant figure in units' place, the value of the number must lie somewhere between 1 and 10. But the logarithm of 1 is 0 and the logarithm of 10 is 1. Hence the logarithm of the proposed number must lie somewhere between 0 and 1. (Art. 3.) That is, its characteristic must be 0.

Thus $\log 2.56 = 0.4082400$

and $\log 9.99 = 0.9995655$

19. THEOREM. *The characteristic of the common logarithm of a number equals the number of places the first significant figure of the number is removed from units' place, and is positive if the first significant figure stands to the left of units' place and is negative if it stands to the right of units' place.*

By the previous article, if the first significant figure stands in units place the characteristic is 0. If the first significant figure stands in the nth place to the left of units place, then the characteristic of its logarithm must be a number such that it can be made from 0 by adding 1 to it n times. (Art. 15.) In other words, the characteristic must be n.

If the first significant figure of the given number stands in the nth place to the right of units place, then the characteristic of its

logarithm must be a number such that it can be made by subtracting 1 from 0 n times; that is, it must be $-n$.

20. EXAMPLES. The above enables us to tell by inspection the characteristic of the common logarithm of any number. Thus the characteristic of the logarithm of 237945.834 is $+5$, because 2, the first significant figure, stands in the fifth place to the left of units place. In the same way, the characteristic of the logarithm of .0007423 is -4, because 7 stands in the fourth place to the right of units place.

In determining the characteristic, care must be taken that we count from the *units place* and *not* from the *decimal point*; for the decimal point stands to the right side of units place.

Find the characteristic of the logarithms of the following numbers:

1.	1888.119	*5.*	3000.0303
2.	.3724	*6.*	.00000000849
3.	783294.009	*7.*	.00010000849
4.	.0084297	*8.*	3.00007

21. TABLES OF COMMON LOGARITHMS. The nature of the work in which logarithms are to be used determines the size and accuracy of the tables which should be employed. For some purposes a table of the logarithms of all numbers from 1 to 10000, given to five or six decimal places, is sufficient. For many purposes a table of the logarithms of all numbers from 1 to 100000 to seven decimal places is desirable, and this may be said to be the standard table. We print herewith a sample page from such a table. This page contains the logarithms of all numbers between 25600 and 26100, or, more correctly, the *mantissas* of the logarithms of these numbers. For, since the characteristic of the common logarithm of any number can always be found by inspection, characteristics are never printed in such tables.

Suppose we wish to find the logarithm of 25964. Then knowing that the characteristic of the logarithm of 25964 is 4, we will find the mantissa from the table. We run down the column headed *Num.* until we come to the figures 2596. We then run across the

A sample page from a table of logarithms:

Num.		0	1	2	3	4	5	6	7	8	9	Diff.	Diff. & Mul.	
25 60	**408**	2400	2569	2739	2909	3078	3248	3417	3587	3757	3926		1	170
61		4096	4265	4435	4604	4774	4944	5113	5283	5452	5622		2	340
62		5791	5961	6130	6300	6469	6639	6808	6978	7147	7317		3	510
63		7486	7656	7825	7994	8164	8333	8503	8672	8841	9011		4	680
64	†	9180	9350	9519	9688	9858	*027	*196	*366	*535	*704		5	850
													6	1020
65	409	0874	1043	1212	1382	1551	1720	1889	2059	2228	2397		7	1190
66		2567	2736	2905	3074	3243	3413	3582	3751	3920	4089		8	1360
67		4259	4428	4597	4766	4935	5105	5274	5443	5612	5781		9	1530
68		5950	6119	6288	6458	6627	6796	6965	7134	7303	7472			
69		7641	7810	7979	8148	8317	8486	8655	8824	8993	9162	169		
25 70	†	9331	9500	9669	9838	*007	*176	*345	*514	*683	*852			
71	410	1021	1190	1359	1527	1696	1865	2034	2203	2372	2541			
72		2710	2878	3047	3216	3385	3554	3723	3891	4060	4229			
73		4398	4567	4735	4904	5073	5242	5410	5579	5748	5917		1	169
74		6085	6254	6423	6592	6760	6929	7098	7266	7435	7604		2	338
													3	507
75		7772	7941	8110	8278	8447	8616	8784	8953	9121	9290		4	676
76	†	9459	9627	9796	9964	*133	*301	*470	*639	*807	*976		5	845
77	411	1144	1313	1481	1650	1818	1987	2155	2324	2492	2661		6	1014
78		2829	2998	3166	3334	3503	3671	3840	4008	4177	4345		7	1183
79		4513	4682	4850	5019	5187	5355	5524	5692	5860	6029		8	1352
													9	1521
25 80		6197	6365	6534	6702	6870	7039	7207	7375	7544	7712			
81		7880	8048	8217	8385	8553	8721	8890	9058	9226	9394			
82	†	9562	9731	9899	*067	*235	*403	*571	*740	*908	1076			
83	412	1244	1412	1580	1748	1917	2085	2253	2421	2589	2757			
84		2925	3093	3261	3429	3597	3765	3933	4101	4269	4437			
												168		
85		4605	4773	4941	5109	5277	5445	5613	5781	5949	6117			
86		6285	6453	6621	6789	6957	7125	7293	7461	7629	7796		1	168
87		7964	8132	8300	8468	8636	8804	8971	9139	9307	9475		2	336
88	†	9643	9811	9978	*146	*314	*482	*649	*817	*985	1153		3	504
89	413	1321	1488	1656	1824	1991	2159	2327	2495	2662	2830		4	672
													5	840
25 90		2998	3165	3333	3501	3668	3836	4004	4171	4339	4507		6	1008
91		4674	4842	5009	5177	5345	5512	5680	5847	6015	6182		7	1176
92		6350	6518	6685	6853	7020	7188	7355	7523	7690	7858		8	1344
93		8025	8193	8360	8528	8695	8863	9030	9197	9365	9532		9	1512
94	†	9700	9867	*035	*202	*369	*537	*704	*872	1039	1206			
95	414	1374	1541	1708	1876	2043	2210	2378	2545	2712	2880			
96		3047	3214	3381	3549	3716	3883	4051	4218	4385	4552			
97		4719	4887	5054	5221	5388	5556	5723	5890	6057	6224			
98		6391	6559	6726	6893	7060	7227	7394	7561	7729	7896			
99		8063	8230	8397	8564	8731	8898	9065	9232	9399	9566		1	167
													2	334
26 00	†	9738	9901	*068	*235	*402	*569	*736	*903	1070	1237	167	3	501
01	415	1404	1570	1737	1904	2071	2238	2405	2572	2739	2906		4	668
02		3073	3240	3407	3574	3741	3907	4074	4241	4408	4575		5	835
03		4742	4909	5075	5242	5409	5576	5743	5909	6076	6243		6	1002
04		6410	6577	6743	6910	7077	7244	7410	7577	7744	7911		7	1169
													8	1336
05		8077	8244	8411	8577	8744	8911	9077	9244	9411	9577		9	1503
06	†	9744	9911	*077	*244	*411	*577	*744	*911	1077	1244			
07	416	1410	1577	1743	1910	2077	2243	2410	2576	2743	2909			
08		3076	3242	3409	3575	3742	3908	4075	4241	4408	4574			
09		4741	4907	5074	5240	5407	5573	5739	5906	6072	6239			
Num.		0	1	2	3	4	5	6	7	8	9	Diff.	Diff. & Mul.	

page on the same horizontal line with 2596 until we come to the column headed 4, at which place will be found the figures 3716, which are the last four figures of the required mantissa. The first three figures are found in the column headed 0 and are seen to be 414. Whence the mantissa of the logarithm of 25964 is .4143716, and therefore

$$\log 25964 = 4.4143716.$$

Of course a decimal point belongs before the mantissa of each logarithm, and since this fact is understood, it is unnecessary to print the decimal points in a table.

Inasmuch as the first three figures of the mantissa are only printed in the column headed 0, it is necessary to mark the point at which these first three figures change. This is done by an asterisk (*) standing in the place of a cipher in the last four figures. Thus to find the logarithm of 25646 we must note that the three figures change from 408 to 409 at the point 25645 (which is indicated by printing *027 in place of 0027), and consequently

$$\log 25646 = 4.4090196.$$

The dagger (†) which appears in column 0 is intended to caution us that the first three figures change at some place in the same horizontal line with it.

If we wish to find the logarithm of a number consisting of more than five figures, say 25705.84, then we must take the nearest number whose logarithm is given in the table, that is to say, 25706.00. Thus

$$\log 25705.84 = 4.4100345, \text{ nearly}.$$

Greater accuracy may be secured by means of tables of differences and multiples, as is explained in connection with any good table of logarithms.

A table of logarithms of numbers from 1 to 100000 can be used to find the logarithm of any number consisting of five significant figures. Thus to find the logarithm of 25.964 we entirely neglect the decimal point in finding the mantissa, as the decimal point affects the characteristic alone. Thus

$$\log 25.964 = 1.4143716.$$

A table of logarithms also enables us to find the number corresponding to any given logarithm by a mere reversal of the process already explained. Thus all numbers the mantissas of whose

logarithms lie between .4082400 and .4166239 are on the specimen page we give. Suppose, as an example, that we wish to find the number corresponding to the logarithm 2.4127469. The characteristic merely affects the decimal point, and consequently the problem is merely to find the significant figures which correspond to the given mantissa. The nearest mantissa printed in the table is .4127461 and this corresponds to the figures 25867. Hence, pointing off the number by means of the characteristic, we find that the number whose logarithm is 2.4127469 is 258.67. In connection with tables of logarithms methods are explained by means of which more figures of this number could be found by means of tables of multiples and differences, or of proportional parts.

22. EXAMPLES.

1. Find the logarithm of 25734.

2. Find the logarithm of 26000000.

3. Find the logarithm of 25.999

4. Find the logarithm of .02578411

5. Find the logarithm of .260099

6. Find the number whose logarithm is 3.4147561

7. Find the number whose logarithm is 0.4104400

8. Find the number whose logarithm is $\bar{2}.415999$

9. Find the number whose logarithm is $\bar{1}.4094094$

10. Find the number whose logarithm is 7.4100000

23. MULTIPLICATION BY LOGARITHMS. Formula (a) (Art. 7) enables us to find the product of several numbers by means of a table of logarithms. Thus, suppose we wish the product of 98 by 265. From a table of logarithms we find

$$\begin{aligned} \log 98 &= 1.9912261 \\ \log 265 &= \underline{2.4232459} \\ \log 98 \times 265 &= 4.4144720 \end{aligned}$$

From the table of logarithms (see sample page) it is found that 4.4144720 is the logarithm of 25970. Therefore $98 \times 265 = 25970$.

24. EXAMPLES IN MULTIPLICATION.

1. Log $327.45 = 2.5151450$ and log $79.493 = 1.9003839$; find the product of 327.45×79.493.

2. Log $.53927 = \bar{1}.7318063$ and log $4.7655 = 0.6781085$; find the product of $.53927 \times 4.7655$.

3. Log $6.3274 = 0.8012253$ and log $1645.6 = 3.2163243$; find the product of 6.3274×1645.6.

25. EXAMPLES IN DIVISION. See Art. 8.

1. Find the quotient of 327.45 by 1645.6.

From a table of logarithms we find

$$\begin{array}{ll} \log 327.45 = & 2.5151450 \\ \log 1645.6 = & \underline{3.2163243} \\ \log 327.45 \div 1645.6 = & \bar{1}.2988207 \end{array}$$

It is seen from a table of logarithms that the number corresponding to the logarithm $\bar{1}.2988207$ is $.19901+$ Therefore $327.45 \div 1645.6 = .19901+$

2. Log $53.927 = 1.7318063$ and log $2.0724 = 0.3164736$; find the quotient of $53.927 \div 2.0724$.

3. Log $33333 = 4.5228744$ and log $13001 = 4.1139768$; find the value of $\frac{33333}{13001}$.

4. Log $54321 = 4.7349678$ and log $20.877 = 1.3196681$; find the value of $54321 \div 20.877$.

26. EXAMPLES IN INVOLUTION. See Art. 9.

1. Find the third power of 1373.3.

From a table of logarithms we find that

$$\begin{array}{lr} & \log 1373.3 = 3.1377654 \\ & \underline{3} \\ \text{whence} & \log (1373.3)^3 = 9.4132962 \end{array}$$

From the sample page it is seen that the number whose logarithm is 9.4132962 equals 2590000000, nearly. Therefore

$$(1373.3)^3 = 2590000000, \text{ nearly.}$$

2. Find the fifth power of 1.9201, whose logarithm is 0.2833238.

3. Find the tenth power of .69353, whose logarithm is 1.8410653.

4. Find the seventh power of 15.926, whose logarithm is 1.2021067.

27. EXAMPLES IN EVOLUTION. See Art. 10.

1. Find the cube root of 26.

From a table of logarithms we find

$$\log 26 = 1.4141374$$

Therefore $$\log \sqrt[3]{26} = 0.4713791$$

The number whose logarithm is 0.4713791 is found to be 2.9606+

Hence $$\sqrt[3]{26} = 2.9606+$$

2. Find the square root of 668.63, whose logarithm is 2.8251859.

3. Find the fifth root of 1109600000000, whose logarithm is 12.0451664.

4. Find the tenth root of 1.384, whose logarithm is 0.1411675.

28. EXPONENTIAL SERIES. The *Exponential Series*, or the *Exponential Theorem*, as it is often called, is an expansion of a^x in terms of the ascending powers of x. The following demonstration* of this important theorem is due to Mr. J. M. Schaeberle, of the Lick Observatory, and is inserted here with his permission.

We are required to expand a^x in a series of ascending powers of x. Assume

$$a^x = A + Bx + Cx^2 + Dx^3 + Ex^4 + \ldots \qquad (1)$$

where A, B, C, etc., are undetermined coefficients.

The limit of the left-hand side of this equation as x approaches 0 is plainly 1. The limit of the right-hand side of this equation as x approaches 0 is A. (See XI, Art. 26.)

Therefore $$A = 1.$$

Substituting this value of A in *(1)* and then squaring both members,

$$a^{2x} = 1 + 2Bx + (2C + B^2)x^2 + (2D + 2CB)x^3 \\ + (2E + 2DB + C^2)x^4 + \ldots \qquad (2)$$

*See *Annals of Mathematics*, Vol. III, p. 153.

But if we substitute $2x$ in place of x in equation *(1)* we obtain

$$a^{2x} = 1 + 2Bx + 4Cx^2 + 8Dx^3 + 16Ex^4 + \ldots \qquad (3)$$

Therefore, equating like powers of x in equations *(2)* and *(3)*, we obtain

$$B = B; \quad C = \frac{B^2}{2}; \quad D = \frac{B^3}{\underline{|3}}; \quad E = \frac{B^4}{\underline{|4}}; \text{ etc.}$$

Whence, on substituting these values of B, C, D, etc., equation *(1)* becomes

$$a^x = 1 + Bx + \frac{B^2x^2}{\underline{|2}} + \frac{B^3x^3}{\underline{|3}} + \frac{B^4x^4}{\underline{|4}} + \ldots \qquad (4)$$

Now, there must exist some quantity, e, at present unknown in value, such that

$$e^B = a, \qquad (5)$$

or, in other words, such that

$$\log_e a = B. \qquad (6)$$

Substituting $\log_e a$ for B throughout equation *(4)* we obtain

$$a^x = 1 + x\log_e a + \frac{x^2(\log_e a)^2}{\underline{|2}} + \frac{x^3(\log_e a)^3}{\underline{|3}} + \frac{x^4(\log_e a)^4}{\underline{|4}} + \ldots \qquad (7)$$

which is called the *Exponential Theorem* or *Series.*

29. TO FIND THE VALUE OF THE BASE e. The base a in the last article is any chosen positive quantity not 1, and its value is therefore at our disposal. Hence in the exponential series (equation 7) we may put $a = e$, so that $\log_e a$ becomes $\log_e e$; that is, 1. Equation *(7)* then becomes

$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \ldots \qquad (1)$$

This important result is convergent for all values of x, (see XIV, Art. 12, Ex. 13,) and consequently the equation is true when $x = 1$. Therefore we have

$$e = 1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \ldots \qquad (2)$$

By taking a sufficient number of terms of this series we may approximate the value of e to any desired degree of accuracy. Thirteen terms of the series give ten places of decimals correctly and we have

$$e = 2.7182818284 \ldots \qquad (3)$$

This number is one of the most important constants in mathematics. It is called the *Naperian Base* and is always represented by the letter e. Its value is known to more than 260 decimal places.

30. LOGARITHMIC SERIES. The *Logarithmic Series* is the expansion of $\log_e(1+x)$ in terms of the ascending powers of x.

From the exponential series

$$a^y = 1 + y\log_e a + \frac{y^2(\log_e a)^2}{\lfloor 2} + \frac{y^3(\log_e a)^3}{\lfloor 3} + \frac{y^4(\log_e a)^4}{\lfloor 4} + \dots \quad (1)$$

Whence, transposing the 1 and dividing through by y,

$$\frac{a^y - 1}{y} = \log_e a + y\left\{\frac{(\log_e a)^2}{\lfloor 2} + \frac{y(\log_e a)^3}{\lfloor 3} + \dots\right\} \quad (2)$$

Therefore, since these variables are always equal,

$$\lim_{y \gtrdot 0}\left\{\frac{a^y - 1}{y}\right\} = \lim_{y \gtrdot 0}\left\{\log_e a + y\left[\frac{(\log_e a)^2}{\lfloor 2} + \frac{y(\log_e a)^3}{\lfloor 3} + \dots\right]\right\} \quad (3)$$

Whence it is easy to see

$$\lim_{y \gtrdot 0}\left\{\frac{a^y - 1}{y}\right\} = \log_e a. \quad (4)$$

Now put $1+x$ for a, then we have

$$\log_e(1+x) = \lim_{y \gtrdot 0}\left\{\frac{(1+x)^y - 1}{y}\right\}$$

Expanding $(1+x)^y$ by binomial formula,

$$\log_e(1+x) = \lim_{y \gtrdot 0}\left\{\frac{y-1}{1.2}x^2 + \frac{(y-1)(y-2)}{1.2.3}x^3 + \frac{(y-1)(y-2)(y-3)}{1.2.3.4}x^4 + \dots\right\} \quad (5)$$

The limit of the right member as $y \gtrdot 0$ can be plainly seen; whence we obtain the equation

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad (6)$$

This is the *Logarithmic Series.*

31. CONVERGENCY OF THE SERIES. The above series is not convergent for values of x greater than 1, and hence cannot be used for computing the logarithm of any integral number but 2. The following scheme will give a series which is available for computing the logarithms of all integers.

32. A LOGARITHMIC SERIES CONVERGENT FOR INTEGRAL VALUES OF x. In the logarithmic series

$$\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ . \ . \ . \qquad (1)$$

Substitute $-x$ for x and we shall have

$$\log_e(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\ . \ . \ . \qquad (2)$$

Subtracting (2) from (1), observing that $\log_e(1+x)-\log_e(1-x)$ $=\log_e\frac{1+x}{1-x}$, we obtain

$$\log_e\frac{1+x}{1-x}=2\left[x+\frac{1}{3}x^3+\frac{1}{5}x^5+\frac{1}{7}x^7+\ . \ . \ .\right] \qquad (3)$$

Now put $x=\frac{1}{2z+1}$, whence $1+x=\frac{2z+2}{2z+1}$, $1-x=\frac{2z}{2z+1}$, and $\frac{1+x}{1-x}=\frac{1+z}{z}$. Therefore we obtain

$$\log_e\frac{1+z}{z}=2\left[\frac{1}{2z+1}+\frac{1}{3(2z+1)^3}+\frac{1}{5(2z+1)^5}+\frac{1}{7(2z+1)^7}+\ . \ .\right] \qquad (4)$$

Whence, since $\log_e\frac{1+z}{z}=\log_e(1+z)-\log_e z$, by substituting and transposing $\log_e z$ we have

$$\log_e(1+z)=\log_e z+2\left[\frac{1}{2z+1}+\frac{1}{3(2z+1)^3}+\frac{1}{5(2z+1)^5}+\ . \ .\right] \qquad (5)$$

This series converges rapidly for integral values of z. Its use in computing the logarithms of numbers will now be explained.

33. TO COMPUTE THE NAPERIAN LOGARITHMS OF NUMBERS. The logarithm of 1 is 0 in all systems. To compute $\log_e 2$, put $z=1$ in equation (5) above. We then obtain

$$\log_e 2=2\left[\frac{1}{3}+\frac{1}{3\cdot 3^3}+\frac{1}{5\cdot 3^5}+\frac{1}{7\cdot 3^7}+\frac{1}{9\cdot 3^9}+\frac{1}{11\cdot 3^{11}}+\ . \ . \ .\right]=.6931472$$

Now put $z=2$ in equation (5). Then we have

$$\log_e 3=.6931472+2\left[\frac{1}{5}+\frac{1}{3\cdot 5^3}+\frac{1}{5\cdot 5^5}+\frac{1}{7\cdot 5^7}+\frac{1}{9\cdot 5^9}+.\right]=1.0986123$$

To find $\log_e 4$ we know $\log_e 4=\log_e 2^2=2\ \log_e 2$; whence

$$\log_e 4=1.3862944$$

To find $\log_e 5$, put $z=4$ in equation (5). We then have

$$\log_e 5 = 1.3862944 + 2\left[\frac{1}{9} + \frac{1}{3\cdot 9^3} + \frac{1}{5\cdot 9^5} + \frac{1}{7\cdot 9^7} + \ldots\right] = 1.6094379.$$

In like manner the logarithms of all numbers may be found. The logarithms of composite numbers need not be computed by the series, since the logarithm of any composite number can be found by adding the logarithms of its component factors.

34. Relation Between the Logarithms of the Same Number in Different Systems.

Consider the systems whose bases are a and e. Then if n is any number, we wish to find the relation between $\log_e n$ and $\log_a n$.

Let $\qquad x=\log_e n$ and $y=\log_a n$.

Then $\qquad n=e^x$ and $n=a^y$;

whence
$$e^x=a^y. \qquad (1)$$

Therefore
$$a=e^{\frac{x}{y}}. \qquad (2)$$

If we write this in logarithmic notation we have

$$\log_e a=\frac{x}{y}, \qquad (3)$$

or, substituting the values of x and y, we obtain

$$\log_e a=\frac{\log_e n}{\log_a n}. \qquad (4)$$

Therefore
$$\log_a n=\frac{1}{\log_e a}\log_e n, \qquad (5)$$

which is the relation between $\log_a n$ and $\log_e n$.

35. Modulus of Common Logarithms. If in equation (5) above we understand e to represent the Naperian base and a the common base, then equation (5) becomes

$$\log n=\frac{1}{\log_e 10}\log_e n. \qquad (1)$$

But $\log_e 10=\log_e 2+\log_e 5=$(by Art. 33) 2.3025851 and $\frac{1}{\log_e 10}$ $=.43429448$ Therefore representing $.43429448$ by M we have

$$\log n=\mathrm{M}\log_e n. \qquad (2)$$

The decimal represented by M is known to 282 decimal places and is called the *Modulus* of the system of common logarithms.

Equation *(2)* is seen to express the important truth that *the common logarithm of any number can be obtained by multiplying the Naperian logarithm of that number by the modulus of the common system.*

36. COMPUTATION OF COMMON LOGARITHMS. We can now compute the common logarithms of numbers. We merely need to multiply each of the Naperian logarithms already found by the modulus .43429448 . . In this manner we find

$$\log 2 = 0.3010300$$
$$\log 3 = 0.4771213$$
$$\log 4 = 0.6020600$$
$$\log 5 = 0.6989700$$
etc. etc.

How can you find log 6?

37. HISTORICAL NOTE. The invention of logarithms is regarded as one of the greatest discoveries in mathematical science. The honor of the invention as well as of the construction of the first logarithmic table belongs to a Scotchman, John Napier (1550–1617), baron of Merchiston. His first work, *Mirifici logarithmorum canonis descriptio*, appeared in 1614 and contained an account of the nature of logarithms (from his standpoint) and a table of natural sines and their logarithms to seven or eight figures. But Napier's logarithms were not the same as those now called Naperian logarithms. The base of his system was not e, although closely related to it.

Henry Briggs, professor of geometry at Gresham College, London, was much interested in Napier's invention and in 1615 visited Napier and suggested to him the advantages of a system of logarithms in which the logarithm of 1 should be 0 and the logarithm of 10 should be 1. Napier, having already thought of the change, gave Briggs every encouragement to compute a system of the new logarithms and made many important suggestions, among which was that of keeping the mantissas of all logarithms positive by using negative characteristics. In 1617 Briggs published the common logarithms of the first 1000 numbers, the book being called *Logarithmorum chilias primi*. Briggs continued to labor at the calculation of logarithms, and in 1624 published his *Arithmetica Logarithmica*, which contained the logarithms of the numbers from 1 to 20000 and from 90000 to 100000 to 14 places of decimals. This gap between 20000 and 90000 was filled up by Adrian Vlacq, who published in 1628 the logarithms of the numbers from 1 to 100000 to ten places. Vlacq's table is the source from which nearly all the tables have been derived which have subsequently been published.

The meaning of logarithms to Napier and Briggs was entirely different from that we now have. They never thought of connecting logarithms with the idea of an exponent, and consequently had no conception of what we call the base of the system. Their idea of logarithm is contained in the meaning of the term itself, which comes from two Greek words meaning *the number of the ratios*. This idea of a logarithm is thus explained: Suppose the ratio of 1 to 10 be divided into a large number of equal ratios (or factors), say 1000000. Then it is true that the ratio of 1 to 2 is composed of 301030 of these equal ratios (or factors), and 301030, *the number of the ratios*, is the *logarithm* of 2. In the same way the ratio of 1 to 3 is composed of 477121 of these equal ratios (or factors), and the *logarithm* of 3 is hence said to be 477121.

The first methods used for computing logarithms were very tedious. The great work of computing was finished long before the discovery of the logarithmic series.

The above note is derived from J. W. L. Glaisher's article on Logarithms in the Encyclopedia Britannica.

THE END.